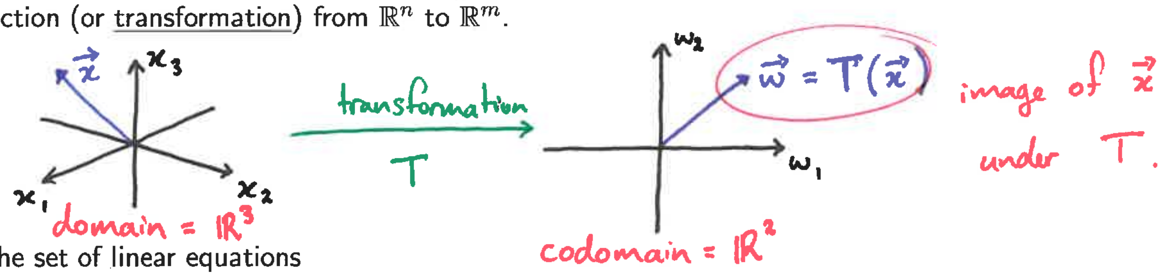
Section 1.8: Introduction to Linear Transformations

Objectives.

- Understand an $m \times n$ matrix as a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Identity the standard basis vectors for $\mathbb{R}^n$ and the standard matrix of a transformation.
- Study some simple linear transformations.

The set of all $n \times 1$ column vectors is denoted by $\mathbb{R}^n$. In this section, we interpret multiplication by an $m \times n$ matrix as a function (or transformation) from $\mathbb{R}^n$ to $\mathbb{R}^m$.

eg.



Example 1. The set of linear equations

$$
\begin{aligned}
w_1 &= x_1 - 2x_2 + 4x_3 - 2x_4 \\
w_2 &= 3x_1 + x_2 - 2x_3 + x_4 \\
w_3 &= -6x_1 + x_3 - x_4
\end{aligned}
$$

defines a linear transformation T_A from $\mathbb{R}^4$ to $\mathbb{R}^3$.

(a) Express the transformation T_A using matrix multiplication.

$$T_A(\vec{x}) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

or: $\vec{w} = T_A(\vec{x}) = A\vec{x}$

where $A = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix}$.

(b) Find the image of the vector $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ under the transformation T_A.

$$T_A\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ -4 \end{bmatrix}$$

Note: The linear transformation in this example can also be written in comma-delimited form as

$$T(x_1, x_2, x_3, x_4) = (\underbrace{x_1 - 2x_2 + 4x_3 - 2x_4}_{w_1}, \underbrace{3x_1 + x_2 - 2x_3 + x_4}_{w_2}, \underbrace{-6x_1 + x_3 - x_4}_{w_3}).$$

Two simple matrix transformations are the zero transformation/operator and the identity transformation/operator.

$T_0(\vec{x}) = 0\vec{x} = \vec{0}$ "zero transformation"

$T_I(\vec{x}) = I\vec{x} = \vec{x}$ "identity transformation"

Properties of matrix transformations. If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and k is a scalar, then:

1. $T_A(\vec{0}) = \vec{0}$ $\longrightarrow$ the zero vector/origin is unchanged by a matrix transformation
2. $T_A(k\vec{u}) = k\,T_A(\vec{u})$ $\rightarrow$ "homogeneity"
3. $T_A(\vec{u}+\vec{v}) = T_A(\vec{u}) + T_A(\vec{v})$ $\rightarrow$ "additive property"

Not all transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are matrix transformations. For instance:

$$\begin{aligned} w_1 &= x_1 + x_2^2 \\ w_2 &= x_1 x_2 \end{aligned}$$

is not a matrix transformation.

"non linear terms"

However, a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies both homogeneity and the additivity property is a matrix transformation.

(More specifically, if these two properties are satisfied then T is called a linear transformation. That is, every matrix transformation is a linear transformation, and every linear transformation is a matrix transformation.)

Example 2. Show that $T(x, y) = (x + 3y, 2x, 2x - y)$ is a linear transformation.

Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$. Then:

$$T(k\vec{u}) = T(ku_1, ku_2) = (ku_1 + 3ku_2,\ 2ku_1,\ 2ku_1 - ku_2)$$
$$= k(u_1 + 3u_2,\ 2u_1,\ 2u_1 - u_2) = k\,T(u_1, u_2) = k\,T(\vec{u}).$$

$$T(\vec{u}+\vec{v}) = T(u_1+v_1, u_2+v_2) = \big(u_1+v_1+3(u_2+v_2),\ 2(u_1+v_1),\ 2(u_1+v_1)-(u_2+v_2)\big)$$
$$= (u_1+3u_2,\ 2u_1,\ 2u_1-u_2) + (v_1+3v_2,\ 2v_1,\ 2v_1-v_2)$$
$$= T(\vec{u}) + T(\vec{v}).$$

T satisfies homogeneity and the additive property, so T is a linear transformation

Theorem. If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and $T_A(\vec{x}) = T_B(\vec{x})$ for every vector $\vec{x}$ in $\mathbb{R}^n$, then $A = B$.

As a consequence of this theorem, each linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ corresponds to exactly one $m \times n$ matrix, which we call the standard matrix for the transformation.

eg. the standard matrix in Ex. 1 is $A = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix}$.

The standard basis vectors for $\mathbb{R}^n$ are the $n \times 1$ vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Every vector in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

eg. in $\mathbb{R}^3$: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$.

Example 3. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$

or: $T(x_1, x_2) = (x_1 - x_2,\ 2x_1 + x_2,\ x_1 + 3x_2)$.

(a) Compute $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2-3 \\ 2(2)+3 \\ 2+3(3) \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 11 \end{bmatrix}$. ← "image of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under T"

(b) Find the image of each standard basis vector in $\mathbb{R}^2$.

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-0 \\ 2(1)+0 \\ 1+3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-1 \\ 2(0)+1 \\ 0+3(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$.

(c) Find the standard matrix for this linear transformation.

$A = \left[\, T(\vec{e}_1) \mid T(\vec{e}_2) \,\right] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$.

Example 4. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}-1\\-2\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = \begin{bmatrix}10\\0\end{bmatrix}.$$

(a) Find the standard matrix for T.

- write $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$.

$$\begin{bmatrix}1\\0\end{bmatrix} = a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}2\\2\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix} = c\begin{bmatrix}1\\-1\end{bmatrix} + d\begin{bmatrix}2\\2\end{bmatrix}$$

- solve for constants.

$$a = \tfrac{1}{2}, \; b = \tfrac{1}{4} \qquad c = -\tfrac{1}{2}, \; d = \tfrac{1}{4}.$$

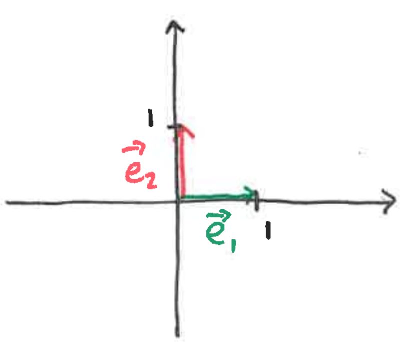
- find $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

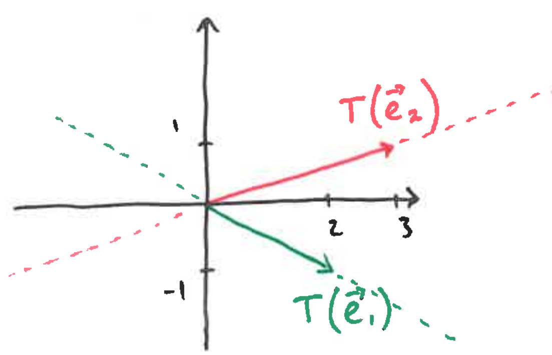
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\tfrac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}2\\2\end{bmatrix}\right) = \tfrac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \tfrac{1}{4}T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix}-1\\-2\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}10\\0\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = -\tfrac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \tfrac{1}{4}T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = -\tfrac{1}{2}\begin{bmatrix}-1\\-2\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}10\\0\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}.$$

$$A = \left[T(\vec{e}_1) \,\middle|\, T(\vec{e}_2)\right] = \begin{bmatrix}2 & 3\\-1 & 1\end{bmatrix}.$$

(b) Sketch a diagram showing each standard basis vector in $\mathbb{R}^2$, and another showing the image of each standard basis vector under the transformation T.

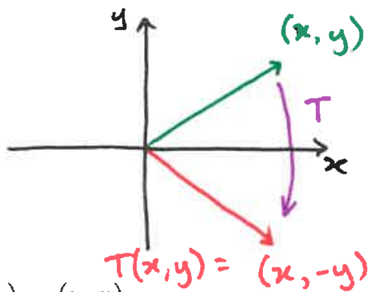




A linear transformation can be interpreted geometrically as a distortion of space that preserves straight lines. (The origin should also remain unchanged!) Some simple examples of these transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ include reflections, (orthogonal) projections, and rotations.

Example 5. For each transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, sketch a diagram showing a typical vector $\vec{x}$ and its image $T(\vec{x})$. Then describe the transformation and find the standard matrix for the transformation.

(a) $T(x, y) = (x, -y)$

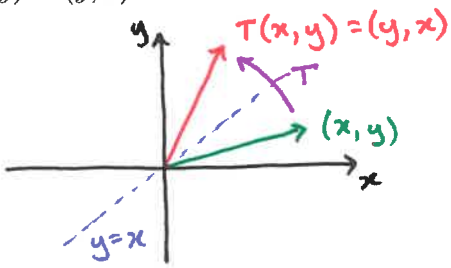


• reflection in x-axis.

$T(1,0) = (1,0)$, $T(0,1) = (0,-1)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) $T(x, y) = (y, x)$

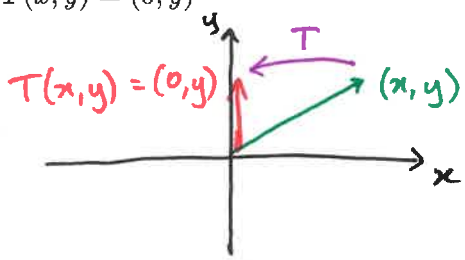


• reflection in the line $y = x$.

$T(1,0) = (0,1)$, $T(0,1) = (1,0)$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) $T(x, y) = (0, y)$



• (orthogonal) projection onto y-axis

$T(1,0) = (0,0)$, $T(0,1) = (0,1)$.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 6. Describe each transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ and find the standard matrix for the transformation.

(a) $T(x, y, z) = (x, 0, z)$

• orthogonal projection onto xz-plane

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

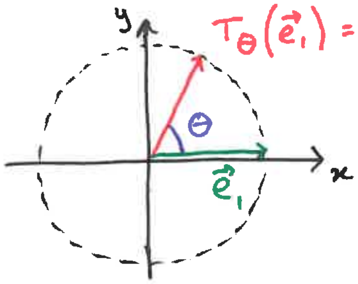
(b) $T(x, y, z) = (x, y, -z)$

• reflection in xy-plane

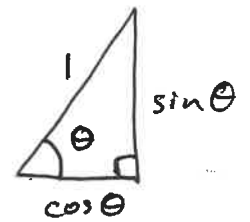
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In $\mathbb{R}^2$, rotation about the origin by an angle θ is a linear transformation. We can find the standard matrix for this rotation by considering the image of the standard basis vectors.

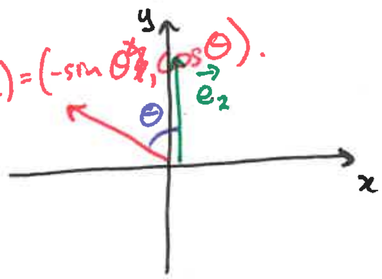
$T_\theta(\vec{e}_1) = (\cos\theta, \sin\theta)$



$$T_\theta(1,0) = (\cos\theta, \sin\theta)$$



$T_\theta(\vec{e}_2) = (-\sin\theta, \cos\theta).$



$$T_\theta(0,1) = (-\sin\theta, \cos\theta).$$

$$\underbrace{R_\theta}_{\text{counterclockwise rotation by } \theta.} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example 7. Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represents a rotation of 45° about the origin.

(a) Find the standard matrix for the transformation.

$$R_\theta = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

(b) Find the image of $\vec{x} = (1, 4)$ under this transformation.

$$T_\theta(1,4) = R_\theta \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} -2.12 \\ 3.54 \end{bmatrix}.$$