Section 1.7: Diagonal, Triangular, and Symmetric Matrices

Objectives.

- Identify diagonal, upper triangular, lower triangular, and symmetric matrices.
- Understand properties of diagonal, triangular, and symmetric matrices.

Some matrices are easier to compute with than others, either because they contain a lot of zeroes or because or their symmetry. These matrices will be important in some of the topics we study later in this course.

A square matrix A is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- diagonal if the only nonzero entries are on the main diagonal.

  i.e. $a_{ij} = 0$ if $i \neq j$

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

- upper triangular if every entry below the main diagonal is zero.

  i.e. $a_{ij} = 0$ if $i > j$

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

- lower triangular if every entry above the main diagonal is zero.

  i.e. $a_{ij} = 0$ if $i < j$

- symmetric if $A = A^T$.

  i.e. $a_{ij} = a_{ji}$

Example 1. Identify each matrix as diagonal and/or upper triangular and/or lower triangular and/or symmetric.

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$$

diagonal, upper Δ, lower Δ, symm.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

symmetric

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ -3 & 8 & 3 \end{bmatrix}$$

lower Δ

$$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

upper Δ

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

none of these!! (not square).

$$\begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

upper Δ

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

symmetric

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

diagonal, upper Δ, lower Δ, symm.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

diagonal, upper Δ, lower Δ, symmetric

An $n \times n$ diagonal matrix can be written in the form

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

This matrix is invertible if and only if every entry on the main diagonal is nonzero, in which case the inverse is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$, provided none of the d_i values is zero.

Example 2. Compute each inverse (if it exists!).

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{-1} =$ DNE!!!

If k is a positive integer, then D^k can be computed by raising each (nonzero) entry in D to the power k.

Example 3. Simplify each expression (if possible!).

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{3} = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & (\frac{1}{3})^3 & 0 \\ 0 & 0 & 1^3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & \frac{1}{27} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-4} = \begin{bmatrix} \frac{1}{16} & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{-4} =$ DNE!!!

cannot find D^{-1}, so we also cannot find D^{-4}.

Multiplication by a diagonal matrix is also relatively simple.

Example 4. Compute each product.

(a) $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 10 \\ 3 & 4 \end{bmatrix}$

i.e. multiply R_1 by -3
multiply R_2 by 2
multiply R_3 by 1

(b) $\begin{bmatrix} 4 & -1 & 2 \\ 8 & 1 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 4 \\ 8 & 4 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

i.e. multiply C_1 by 1
" C_2 by 4
" C_3 by 2

Properties of (upper) triangular matrices. *Note: similar properties hold for lower triangular matrices.*

1. The transpose of an upper triangular matrix is lower triangular.
2. The product of two upper triangular matrices is upper triangular.
3. An upper triangular matrix is invertible if and only if every entry on the main diagonal is nonzero.
4. The inverse of an invertible upper triangular matrix is upper triangular.

Example 5. Suppose that $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Show that $A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } A^{-1} = \begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}.$$

(b) Compute AB and BA. What do you notice?

$$AB = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

Proof of 2. Suppose A, B are upper triangular, and let $C = AB$.
If $i > j$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$
$$= \underbrace{a_{i1}b_{1j} + \cdots + a_{i(i-1)}b_{(i-1)j}}_{a_{i1}=0,\ \ldots,\ a_{i(i-1)}=0} + \underbrace{a_{ii}b_{ij} + \cdots + a_{in}b_{nj}}_{b_{ij}=0,\ \ldots,\ b_{nj}=0}$$
$$= \underline{0}.$$

Because $c_{ij} = 0$ when $i > j$, the matrix C is upper triangular.

A symmetric $\iff A = A^T$

Properties of symmetric matrices. If A and B are symmetric $n \times n$ matrices, and k is a scalar, then:

1. A^T is symmetric.
2. $A + B$ and $A - B$ are both symmetric.
3. kA is symmetric
4. AB is symmetric if and only if $AB = BA$.
5. If A is invertible then A^{-1} is symmetric.
6. If A is invertible, then AA^T and $A^T A$ are invertible.

Proof of 2.

$$(A+B)_{ij} = A_{ij} + B_{ij} = A_{ji} + B_{ji} = (A+B)_{ji}.$$

row i, col. j of $A+B$

The next example illustrates property 4 above.

Example 6. Compute each product.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ } AB is symmetric, so $AB = BA$.

(d) $\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

One final observation is that for <u>any</u> matrix A, the products AA^T and $A^T A$ are both symmetric.

$$(AA^T)^T = (A^T)^T A^T = AA^T, \text{ so } AA^T \text{ is symmetric.}$$

Swap order when using transpose!!!

Example 7. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix}$. Confirm that both AA^T and $A^T A$ are symmetric.

$$AA^T = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 19 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 3 & 7 \\ 3 & 1 & 3 \\ 7 & 3 & 10 \end{bmatrix}.$$