Section 1.6: More on Linear Systems and Invertible Matrices

Objectives.

- Use an inverse matrix to solve a linear system.
- Understand properties of invertible matrices.
- Determine all vectors $\vec{b}$ for which the linear system $A\vec{x} = \vec{b}$ is consistent.

Theorem. A linear system has either no solutions, exactly one solution, or an infinite number of solutions.

Proof. Suppose $\vec{x}_1$ and $\vec{x}_2$ are distinct solutions of $A\vec{x} = \vec{b}$.

Let $\vec{x}_0 = \vec{x}_1 - \vec{x}_2$. Then $\vec{x}_0 \neq \vec{0}$ because $\vec{x}_1 \neq \vec{x}_2$. Also:

$$A\vec{x}_0 = A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}.$$

If k is any scalar, then:

$$A(\vec{x}_1 + k\vec{x}_0) = A\vec{x}_1 + kA\vec{x}_0 = \vec{b} + k\cdot\vec{0} = \vec{b} + \vec{0} = \vec{b}.$$

That is, $\vec{x}_1 + k\vec{x}_0$ is a sol$^{\text{n}}$ of $A\vec{x} = \vec{b}$ for any k.

Therefore, this system has infinitely many solutions.

Theorem. If A is an invertible $n \times n$ matrix, and $\vec{b}$ is an $n \times 1$ column vector, then the linear system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

From the previous theorem, if A is invertible then the system $A\vec{x} = \vec{b}$ can be solved by multiplying by A^{-1}.

Example 1. Solve the linear system.

$$\begin{aligned} 6x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + x_3 &= 0 \\ 10x_1 + 3x_2 + 4x_3 &= -1 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

The inverse of $A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$.

Thus:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ 4 \end{bmatrix}.$$

Sometimes we may want to solve several linear systems that have the same coefficient matrix A. For instance, supose that we want to solve all of the systems:

$$A\vec{x} = \vec{b}_1, \quad A\vec{x} = \vec{b}_2, \quad \dots, \quad A\vec{x}_k = \vec{b}_k.$$

If A is invertible, then the solutions can be found using matrix multiplication.

$$\vec{x}_1 = A^{-1}\vec{b}_1, \quad \vec{x}_2 = A^{-1}\vec{b}_2, \quad \dots, \quad \vec{x}_k = A^{-1}\vec{b}_k$$

An alternate approach (which also works when A is singular!) is to solve the systems at the same time by row-reducing the augmented matrix

$$\left[A \,\middle|\, \vec{b}_1 \,\middle|\, \vec{b}_2 \,\middle|\, \cdots \,\middle|\, \vec{b}_k \right] \xrightarrow{\text{row operations}} \left[I \,\middle|\, \vec{x}_1 \,\middle|\, \vec{x}_2 \,\middle|\, \cdots \,\middle|\, \vec{x}_k \right].$$

Example 2. Solve the linear systems.

(a)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ x_2 - 2x_3 &= -2 \\ 2x_1 - 3x_2 + 2x_3 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 1 \\ x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 + 2x_3 &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 2 & -3 & 2 & 4 & -1 \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 3 & -6 & -6 & -3 \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

$$\Big\downarrow \; R_1 \to R_1 + 3R_2$$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & -2 & -1 & 4 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

part (a): $x_3 = t$, so
$x_1 = -1 + 2t$, $x_2 = -2 + 2t$.

part (b): system is inconsistent, so there are no solutions.

Our *definition* of an inverse matrix $B = A^{-1}$ requires that both $AB = I$ and $BA = I$ are true. However, it is enough to know that at least one of these equations is true.

Theorem. Let A and B be a square matrices. If $AB = I$ or $BA = I$, then $B = A^{-1}$.

Example 3. Show that $B = A^{-1}$ for the matrices A and B below. (These are the matrices from Example 1.)

$$A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

From the Thm above, we only need to show $AB = I$ (or $BA = I$).

$$AB = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Therefore, $B = A^{-1}$. (also, $A = B^{-1}$).

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.

from Sect. 1.5 (page 2)

5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.

new conditions.

Theorem. If A and B are square matrices and AB is invertible, then both A and B are invertible.

Proof. Suppose $\vec{x}_0$ is a sol^n to $B\vec{x} = \vec{0}$. Then

$$(AB)\vec{x}_0 = A(B\vec{x}_0) = A\vec{0} = \vec{0}.$$

Because AB is invertible, the system $(AB)\vec{x} = \vec{0}$ has only the trivial solution. Thus $\vec{x}_0 = \vec{0}$. That is, $B\vec{x} = \vec{0}$ has only the trivial solution, so B is invertible.

Thus $A = A(BB^{-1}) = \underline{(AB)B^{-1}}$ is invertible.

product of invertible matrices is invertible.

Problem. Given an $m \times n$ matrix A, find all $m \times 1$ vectors $\vec{b}$ for which the linear system $A\vec{x} = \vec{b}$ is consistent.

If A is invertible, this problem is easy. ($A\vec{x} = \vec{b}$ is consistent for every $m \times 1$ vector $\vec{b}$.) Otherwise, row operations can be used to determine which vectors $\vec{b}$ give consistent systems.

Example 4. What conditions must b_1, b_2, b_3 satisfy for the system below to be consistent?

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= b_1 \\ x_2 - 2x_3 &= b_2 \\ 2x_1 - 3x_2 + 2x_3 &= b_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 2 & -3 & 2 & b_3 \end{array}\right]$$

$$\Big\downarrow\ R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 3 & -6 & b_3 - 2b_1 \end{array}\right]$$

$$\Big\downarrow\ R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 - 3b_2 \end{array}\right]$$

For this system to be consistent, we need

$b_3 - 2b_1 - 3b_2 = 0$, so $b_3 = 2b_1 + 3b_2$.