Section 1.5: Elementary Matrices and a Method for Finding A^{-1}

Objectives.

- Write each elementary row operation using matrix multiplication.
- Find the inverse of a given row operation.
- Use row operations to find the inverse of a matrix or show that the matrix is not invertible.

Recall the three elementary row operations:

- multiply one row by a constant
- swap two rows
- add a multiple of one row to another row.

Two matrices A and B are row equivalent if A can be transformed into B using elementary row operations.

eg. $\begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are row equivalent.

(i.e) $R_1 \leftrightarrow R_2$, then $R_1 \to \frac{1}{4} R_1$

An elementary matrix is a matrix that can be obtained from an identity matrix using a single elementary row operation. Multiplication by an elementary matrix is the same as performing an elementary row operation.

Example 1. What elementary row operation is equivalent to calculating EB for each matrix E below?

(a) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← double row 2 of B.

$R_2 \longrightarrow 2R_2$

(c) $E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Swap row 1 and row 3.

$R_1 \longleftrightarrow R_3$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← add -3 times row 1 to row 2

$R_2 \to R_2 - 3R_1$

eg. $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

(d) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← multiply row 1 by 1.

$R_1 \to R_1$

Each elementary row operation can be reversed by applying another elementary row operation.

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 5R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2. Find an elementary 3×3 matrix that corresponds to each row operation, and find an elementary row operation that reverses each row operation.

	elementary matrix	<u>inverse</u> row operation
(a) multiply row 3 by $-\frac{1}{5}$ i.e. $R_3 \to -\frac{1}{5}R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{5} \end{bmatrix}$	$R_3 \to -5R_3$ i.e. multiply by reciprocal.
(b) swap row 1 and row 2 i.e. $R_1 \leftrightarrow R_2$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$ i.e. swap rows again!!!
(c) add 4 times row 2 to row 1 i.e. $R_1 \to R_1 + 4R_2$	$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \to R_1 - 4R_2$ i.e. ~~substract~~ subtract $4R_2$ from R_1.

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution. ← note: $A\vec{x} = \vec{0}$ always has the solution $\vec{x} = \vec{0}$. If A is invertible, then $\vec{x} = \vec{0}$ is the only solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.

The Equivalence Theorem says that if A is invertible then there is a sequence of elementary row operations that reduces A to I_n. The same sequence of row operations applied to I_n results in the matrix A^{-1}.

Inverting a matrix. To find the inverse of an $n \times n$ matrix A:

1. Form the matrix $[A|I_n]$.
2. Apply elementary row operations to reduce A to I_n.
3. The resulting matrix has the form $[I_n|A^{-1}]$.

Example 3. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

• start with $[A \mid I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\downarrow R_3 \to -R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\downarrow \begin{array}{l} R_2 \to R_2 + 3R_3 \\ R_1 \to R_1 - 3R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\downarrow R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

... finish with $[I \mid A^{-1}]$.

Therefore:

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

The algorithm for finding an inverse matrix can also be used to decide whether a matrix has an inverse.

Example 4. Determine whether $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 3 & 5 & 11 \end{bmatrix}$ is invertible and find the inverse if possible.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 5 & 11 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 10 & -2 & 1 & 0 \\ 0 & 2 & 20 & -3 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 - 2R_2$$

uh-oh!!! $\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 10 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$

A is not invertible, because we cannot reduce A to I_3 using elementary row operations.

Example 5. Decide whether each homogeneous linear system has nontrivial solutions.

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad\;\; + 8x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is invertible (Ex. 3), so the system has only the trivial solution.

(b)
$$\begin{aligned} x_1 + x_2 - 3x_3 &= 0 \\ 2x_1 + 3x_2 + 4x_3 &= 0 \\ 3x_1 + 5x_2 + 11x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 3 & 5 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The coefficient matrix is not invertible (Ex. 4), so there are nontrivial solutions.