Section 1.4: Inverses; Algebraic Properties of Matrices

Objectives.

- Learn the algebraic rules for matrix addition and multiplication.
- Understand zero matrices, identity matrices, and inverse matrices.
- Find the inverse of a 2×2 matrix.
- Use an inverse matrix to solve a linear system.
- Compute powers of matrices and matrix polynomials.

Many of the rules for matrix algebra will be familiar from previous mathematics classes.

Properties of matrix algebra. Lower case letters refer to scalars; upper case letters refer to matrices.

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A(BC) = (AB)C$
4. $A(B \pm C) = AB \pm AC$
5. $(A \pm B)C = AC \pm BC$
6. $a(B \pm C) = aB \pm aC$
7. $(a \pm b)C = aC \pm bC$
8. $a(bC) = (ab)C$
9. $a(BC) = (aB)C = B(aC)$

Notice however that **matrix multiplication is <u>not</u> commutative**. That is, $AB \neq BA$ in general.

Example 1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. Compute AB and BA.

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

not equal!!!

$$BA = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{2\times 2}$$

The $m \times n$ matrix where every entry is 0 is a zero matrix and is denoted by $0_{m\times n}$.

eg. $0_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $0_{1\times4} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

Properties of zero matrices.

1. $A \pm 0 = A$
2. $A - A = 0$
3. $\underset{\text{scalar}}{0}A = \underset{\text{matrix}}{0}$
4. If $cA = 0$, then either $c=0$ or $A=0$.

The last property listed above is called the zero-product principle. This is **not** true for matrix multiplication, as shown in the previous example.

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0_{2\times2},$$ but neither factor is zero.

It is also incorrect to 'cancel' factors in a matrix product.

Example 2. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 3 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 3 \\ -1 & -2 \end{bmatrix}$. Compute AB and AC.

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Thus $AB = AC$, but $B \neq C$.

A square matrix with 1 on the main diagonal and 0 everywhere else is called an identity matrix. This is denoted by either I or I_n (to specify the size of the matrix).

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.

Properties of identity matrices. Let A be an $m \times n$ matrix.

1. $AI_n = A$
2. $I_mA = A$

Example 3. Confirm the properties above for the matrix $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$.

$$AI = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = A.$$

If A is a square matrix, and B is a square matrix such that $AB = BA = I$, then we call A an invertible matrix (or nonsingular matrix) and we call B an inverse of A.

Example 4. Show that $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

$$AB = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$AB = I$ and $BA = I$, so $B = A^{-1}$.

If A does not have an inverse, then A is not invertible (or singular).

Example 5. Show that $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ is a singular matrix.

guess: $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Suppose $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus:
$$\begin{aligned} 1a + 0c &= 1 \Rightarrow a = 1 \\ 1b + 0d &= 0 \Rightarrow b = 0 \\ 2a + 0c &= 0 \Rightarrow a = 0 \end{aligned}$$

contradiction!!!

We have found a contradiction, so A has no inverse.

Example 6. Show that if B and C are both inverses of A, then $B = C$.

By assumption, $AB = I$ and $CA = I$. Then:

$$B = IB = (CA)B = C(AB) = CI = C.$$

The previous example shows that if A is invertible then its inverse is unique. We denote this inverse by A^{-1}.

eg. $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$. (from Ex. 4).

Example 7. Show that if A and B are both invertible and have the same size, then $(AB)^{-1} = B^{-1}A^{-1}$.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Thus AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(The quantity $ad - bc$ is called the <u>determinant</u> of A. We study determinants in Chapter 2.)

Example 8. Decide whether each matrix is invertible, and find the inverse if possible.

(a) $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$\det(A) = (6)(1) - (-3)(-2) = 6 - 6 = 0$

A is <u>not</u> invertible. (ie. A is singular)

(b) $B = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}$

$\det(B) = (5)(1) - (1)(3) = 5 - 3 = 2.$

$\det B \neq 0$, so B has an inverse!!!

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}.$$

Recall that a linear system can be written in the form $A\vec{x} = \vec{b}$. If the coefficient matrix A is invertible, then the linear system can be solved by <u>multiplying both sides of the matrix equation by A^{-1}</u>.

Example 9. Solve the linear system.

$$5x + y = 2$$
$$3x + y = -2$$

$$\begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -8 \end{bmatrix}.$$

i.e. $x = 2$, $y = -8$.

if $A\vec{x} = \vec{b}$, then

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

A square matrix can be raised to any nonnegative integer power.

$$A^0 = I, \quad A^1 = A, \quad A^2 = AA, \quad A^3 = AAA, \ldots$$

An invertible matrix can be raised to any integer power (positive or negative).

$$A^{-n} = (A^{-1})^n.$$

Powers of invertible matrices. Let A be invertible, n be an integer, and k be a nonzero scalar.

1. A^{-1} is invertible, and $(A^{-1})^{-1} = A$
2. A^n is invertible, and $(A^n)^{-1} = (A^{-1})^n$
3. kA is invertible, and $(kA)^{-1} = k^{-1}A^{-1} = \frac{1}{k}A^{-1}$.

If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a polynomial and A is a square matrix, then

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

Example 10. Let $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ and let $p(x) = x^2 - x + 3$.

(a) Compute A^3.

$$A^3 = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}.$$

(b) Compute $p(A)$.

$$p(A) = A^2 - A + 3I = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 8 & 3 \end{bmatrix}.$$

Recall that the transpose of a matrix is found by swapping the rows and the columns of the matrix.

Example 11. Show that if A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

$$A^T (A^{-1})^T = (AA^{-1})^T = I^T = I.$$

$$(A^{-1})^T A^T = (A^{-1}A)^T = I^T = I.$$

Therefore, $(A^T)^{-1} = (A^{-1})^T$.