Section 1.3: Matrices and Matrix Operations

Objectives.

- Recognize a rectangular array of numbers as a matrix.
- Understand basic terminology and notation used for matrices.
- Apply the operations of matrix addition, subtraction, and multiplication correctly.
- Compute a linear combination of matrices.
- Find the transpose and the trace of a matrix.

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns. A square matrix of order n is a matrix with n rows and n columns.

"main diagonal"

eg. $\begin{bmatrix} 1 & 3 & 2 & 0 \\ 2 & 2 & 1 & 2 \\ 1 & -2 & 4 & 6 \end{bmatrix}$ is a 3×4 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a ~~ma~~ square matrix of order 2.

A matrix with one row is called a row vector (or row matrix). A matrix with one column is called a column vector (or column matrix).

row vector: $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ column vector: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Two matrices are equal if they have the same size and their corresponding entries are equal. If two matrices have the same size, then their sum (or difference) is found by adding (or subtracting) corresponding entries. A matrix can be multiplied by a scalar by multiplying each entry by the scalar.

Example 1. Simplify each expression.

(a) $$\begin{bmatrix} 3 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 \\ 4 & 2 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 3 & 2 & 1 \\ 1 & -5 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 & 7 \\ 1 & 2 & 3 & 0 \\ 5 & -3 & 9 & 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 7 & 3 & 0 & 2 \\ 5 & -1 & 2 & \cancel{-}1 \\ -2 & 2 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & -5 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & 1 \\ 5 & -2 & 0 & \cancel{-3} \\ -3 & 7 & -6 & -4 \end{bmatrix} -1$$

(c) $$2\begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -4 \\ 2 & -2 & 0 \\ -6 & 4 & 8 \end{bmatrix}$$

different sizes!!!

note: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is undefined!!!

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is an $m \times n$ matrix. The entry in the ith row and jth column of AB is found by multiplying each entry in the ith row of A by the corresponding entry in the jth column of B and adding the results.

$$\begin{array}{cc} & \\ & \\ i\text{th row} \rightarrow & \\ & \\ & \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{array}{c} \quad\quad\quad\quad \downarrow j\text{th column} \\ \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & \vdots & & \\ b_{r1} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \end{array}$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

Example 2. Compute each product below (if possible).

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 \\ -2 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 6 \\ 9 & 17 & 7 \end{bmatrix}$

$(1)(3) + (2)(-2) + (-1)(1) = -2$ $\quad$ $(1)(4) + (2)(2) + (-1)(3) = 5$

$(2)(3) + (0)(-2) + (3)(1) = 9$ $\quad$ $(2)(4) + (0)(2) + (3)(3) = 17$

etc.

(b) $\begin{bmatrix} 3 & 4 & -1 \\ -2 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix} =$ undefined.

$3 \times \underline{3}$ matrix $\quad$ $\underline{2} \times 3$ matrix

dimensions do <u>not</u> match

(c) $\begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 & 5 \\ 3 & 2 & 1 & -4 \\ -3 & 0 & 3 & 6 \end{bmatrix}$

A matrix can be partitioned into submatrices by selecting certain rows and/or columns.

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{where} \quad A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}, \quad A_{21} = \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} a_{33} \end{bmatrix}$$

$$\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix} \quad \text{where} \quad \vec{r}_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}, \text{ etc.}$$

$$\left[\begin{array}{c|c|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{bmatrix} \quad \text{where} \quad \vec{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \text{ etc.}$$

Partitioning matrices into rows and columns allows some different strategies for matrix multiplication. This is particularly useful when only some rows and/or columns of the product are needed.

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$$

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} B = \begin{bmatrix} \vec{a}_1 B \\ \vec{a}_2 B \\ \vdots \\ \vec{a}_n B \end{bmatrix}$$

Example 3. Simplify each expression.

(a) $$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

compare with last column of Ex. 2(a).

(b) $$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -3 \end{bmatrix} = 1\begin{bmatrix} -1 & 0 & 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & -4 \end{bmatrix}.$$

compare with 2nd row of Ex. 2(c).

If $A_1, A_2, \dots, A_n$ are matrices of the same size, and $c_1, c_2, \dots, c_n$ are scalars, then

$$c_1 A_1 + c_2 A_2 + \cdots + c_n A_n$$

is a linear combination of $A_1, A_2, \dots, A_n$.

When B is a column vector, the product AB is a linear combination of the columns of A.

Example 4. Simplify.

$$\begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y + z \\ -3x + 4y \\ -x + 8y + 5z \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} z$$

The last example suggests that we can express a linear system using matrix multiplication rather than an augmented matrix.

- linear system:

$$\begin{aligned} 2x + y + z &= 5 \\ -3x + 4y &= 2 \\ -x + 8y + 5z &= 0 \end{aligned}$$

equivalent ways of writing a linear system

- augmented matrix:

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ -3 & 4 & 0 & 2 \\ -1 & 8 & 5 & 0 \end{bmatrix}$$

- matrix equation:

$A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

If A is an $m \times n$ matrix, then the transpose A^T is the $n \times m$ matrix is obtained by swapping the rows and columns of A.

Example 5. Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 & 2 & 3 \\ -5 & 1 & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & -5 \\ 2 & 1 \\ 3 & 6 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

$B^T = \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$C^T = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}$

(d) $D = \begin{bmatrix} 3 & 4 & 0 \\ 4 & 5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

$D^T = \begin{bmatrix} 3 & 4 & 0 \\ 4 & 5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

note: $D^T = D$, so D is symmetric

Properties of transposes.

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$ note that the order is swapped!!!

The trace of a square matrix A, denoted by $\mathrm{tr}(A)$, is the sum of the entries on the main diagonal. (The trace is undefined for matrices that are not square.)

Example 6. Find the trace (if possible) of each matrix in the previous example.

tr A and tr B are undefined, because A and B are not square matrices.

$\mathrm{tr}(C) = 1+1+1 = 3$, $\mathrm{tr}(D) = 3+5+(-1) = 7.$