Section 1.2: Gaussian Elimination

Objectives.

- Identify matrices in row echelon form and reduced row echelon form.
- Use an augmented matrix in reduced row echelon form to write the solution for a linear system.
- Apply Gauss-Jordan elimination and Gaussian elimination to solve a linear system.
- Understand the relationship between numbers of unknowns, equations, and free variables.

A matrix is in row echelon form when the following are true.

(a) If a row contains a nonzero number, then the first nonzero number in the row is a 1. (This is a leading 1.)

(b) Any rows that contain only zeroes are at the bottom of the matrix.

(c) If a row has a leading 1, then it is further to the right than the leading 1 in any higher row.

A matrix is in reduced row echelon form if it is in row echelon form and:

(d) If a column contains a leading 1, then every other number in the column is 0.

Example 1. Which of the matrices below are in row echelon form (ref)? Which are in reduced row echelon form (rref)? Which are neither?

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

r.e.f.

swap
$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -7 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

r.r.e.f.

needs to be 1
$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

neither

swap
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

r.r.e.f.

$$\begin{bmatrix} 1 & -4 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

r.e.f.

should be 1
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

neither

swap
$$\begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 2 & 4 & 0 & 8 \\ 1 & 0 & -5 & 2 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

r.r.e.f.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 7 \\ 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

r.r.e.f.

A variable corresponding to the leading 1 in some row is a leading variable. All other variables are free variables.

Example 2. Each augmented matrix below is in reduced row echelon form, and corresponds to a linear system in the variables x, y, and z. Find a solution for each linear system, identify the leading variables and the free variables, and describe the solution geometrically.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

leading 1s

leading vars: x, y, z free vars: none

Solution is $x=1$, $y=2$, $z=3$.

This is the point $(1,2,3)$.

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 4 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free variable

leading vars: x, y free vars: z

Let $z = t$. ← assign a parameter to each free var.

$x + 4t = -3 \Rightarrow x = -4t - 3$

$y + 2t = 0 \Rightarrow y = -2t$

The solution is $x = -4t-3$, $y = -2t$, $z = t$.

This is a line in three-dimensional space.

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

inconsistent system!!

$0x + 0y + 0z = 1$.

leading vars: x, y free vars: z

System has no solution.

leading 1

(d) $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free variables

leading vars: x free vars: y, z

Let $y = s$, $z = t$.

Then $x - s - t = 0$, so $x = s + t$.

The sol$^{\underline{n}}$ is $x = s+t$, $y = s$, $z = t$.

This sol$^{\underline{n}}$ is a plane in three-dimensions.

Given an augmented matrix, an algorithm called Gaussian elimination can be used to find a matrix in row echelon form that has the same solutions.

Gaussian elimination.

1. Identify the leftmost column that contains a nonzero number.
2. If necessary, swap two rows so that the first number in this column is nonzero. Call this number a.
3. Multiply the top row by $\frac{1}{a}$ to create a leading 1.
4. Add multiples of the top row to each lower row so that every entry below the leading 1 is zero.
5. Cover the top row and repeat from Step 1.

Example 3. Apply Gaussian elimination to the augmented matrix below.

need a 1 at the top.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow R_1 \to \tfrac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\vdots$$

$$\downarrow R_2 \to -\tfrac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - 5R_2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

$$\downarrow R_3 \to 2R_3$$

$$\underbrace{\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{\text{matrix is in row echelon form.}}$$

leading vars. are x_1, x_3, x_5.

free vars. are x_2, x_4.

While Gaussian elimination will result in a matrix in row echelon form, Gauss-Jordan elimination is an extension that gives a matrix in reduced row echelon form.

> **Gauss-Jordan elimination.**
>
> 1. Perform Gaussian elimination to obtain a matrix in row echelon form.
> 2. Starting from the bottom row and working upwards, identify the leading 1 in each row (if there is one).
> 3. Add multiples of this row to each higher row so that each entry above the leading 1 is a zero.

Example 4. Solve the linear system.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 + 15x_6 &= 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$$\Big\downarrow \; \begin{aligned} R_2 &\to R_2 - 2R_1 \\ R_4 &\to R_4 - 2R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\Big\downarrow \; R_2 \to -R_2$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 6 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\Big\downarrow \; \begin{aligned} R_3 &\to R_3 - 5R_2 \\ R_4 &\to R_4 - 4R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$$\Big\downarrow \; R_3 \leftrightarrow R_4$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow \; R_3 \to \tfrac{1}{6} R_3$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow \; R_2 \to R_2 - 3R_3$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow \; R_1 \to R_1 + 2R_2$$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_2 = r$, $x_4 = s$, $x_5 = t$. Then:

$$x_1 = -3r - 4s - 2t$$

$$x_3 = -2s$$

$$x_6 = 1/3.$$

A linear system is homogeneous if each of the equations in the system is homogeneous.

i.e.
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

A homogeneous linear system in the variables $x_1, x_2, \ldots, x_n$ always has the trivial solution

$$x_1 = x_2 = \cdots = x_n = 0.$$

(Any solution where at least one variable is nonzero is called a nontrivial solution.)

Example 5. Solve the linear system. *Hint: compare this system with the previous example.*

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right]$$

elementary row operations do not affect a column of zeros!!

$\downarrow$ row operations from Ex. 4.

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

r.r.e.f. from Ex. 4.

Let $x_2 = r$, $x_4 = s$, $x_5 = t$.

Then:

$x_1 = -3r - 4s - 2t$

$x_3 = -2s$

$x_6 = 0$

note: if $r = s = t = 0$, the solution is $(0,0,0,0,0,0)$.

trivial sol$^{\underline{n}}$.

Theorem. A homogeneous linear system with n unknowns and r nonzero rows in the reduced row echelon form of the augmented matrix has $n - r$ free variables.

Theorem. A homogeneous linear system with more unknowns that equations has infinitely many solutions.

An alternative to Gauss-Jordan elimination is to use Gaussian elimination followed by back-substitution.

Gaussian elimination with back-substitution.

1. Perform Gaussian elimination to obtain a matrix in row echelon form.
2. Write an equation for each leading variable in terms of the other variables.
3. Starting from the bottom, substitute each equation into the equations above it.
4. Replace each free variable with a parameter.

Example 6. Use back-substitution to solve the linear system in Example 4.

From Ex. 4.

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ x_3 + 2x_4 + 3x_6 = 1 \\ x_6 = 1/3 \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2x_3 - 2x_5 \\ x_3 = 1 - 2x_4 - 3x_6 \\ x_6 = 1/3 \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2x_3 - 2x_5 \\ x_3 = 1 - 2x_4 - 3\left(\frac{1}{3}\right) = -2x_4 \\ x_6 = \frac{1}{3} \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2(-2x_4) - 2x_5 \\ \quad = -3x_2 - 4x_4 - 2x_5 \\ x_3 = -2x_4 \\ x_6 = 1/3 \end{cases}$$

The solution is:

$x_1 = -3r - 4s - 2t$
$x_2 = r$
$x_3 = -2s$
$x_4 = s$
$x_5 = t$
$x_6 = \frac{1}{3}$.

Discussion. For each augmented matrix below, identify the number of solutions for the corresponding linear system.

$$\left[\begin{array}{cccc|c} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variable, system is consistent $\Rightarrow$ infinite solutions

$$\left[\begin{array}{cccc|c} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

no free variable, consistent system $\Rightarrow$ one solution.

$$\left[\begin{array}{cccc|c} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$0 = 1$!!!!

no solutions (inconsistent system)