Section 1.1: Introduction to Systems of Linear Equations

Objectives.

- Identify linear and nonlinear equations, and systems of linear equations.
- Understand terminology related to linear systems and matrices.
- Solve simple linear systems and interpret their solutions geometrically.
- Introduce elementary row operations.

A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where not all the a_i are zero.

A homogeneous linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, where not all the a_i are zero.

Example 1. Underline the linear equations. Circle the homogeneous linear equations.

$x + 4y = 9$	$w + 3x - y^2 + z = 3$ not linear	$-3x + 2y - \frac{1}{2}z = 0$
$x_1 - \sqrt{x_2} = 0$ not linear	$4x_1 - 2x_2 + 3x_3 = 0$	$x_1 + x_2 + x_3 + x_4 = 1$

A finite set of linear equations is called a system of linear equations (or linear system). The variables are called the unknowns.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

A solution of a linear system is an assignment of a number to each unknown so that each equation in the linear system is true.

Example 2. Decide whether each set of numbers is a solution to the linear system below.

$$\begin{aligned} x + y + 3z &= 0 \\ 2x + y - z &= 5 \end{aligned}$$

(a) $x = 0$, $y = 0$, $z = 0$

$0 + 0 + 3(0) = 0$

$2(0) + 0 - 0 = 0 \neq 5$

not a solution!!!

(b) $x = 5$, $y = -5$, $z = 0$

$5 + (-5) + 3(0) = 0$

$2(5) + (-5) - 0 = 5$

this is a solution!!

(c) $x = 1$, $y = 2$, $z = -1$

$1 + 2 + 3(-1) = 0$

$2(1) + 2 - (-1) = 5$

this is a solution!!

The set of solutions of a linear equation in x and y is a line in the xy-plane, so a solution of a linear system in x and y corresponds to a point of intersection between lines.

Example 3. Solve each linear system, and interpret the solution(s) geometrically.

(a) $$x + y = 1$$
$$2x + y = 4$$

• add $-2 \times$ eq 1 to eq 2:

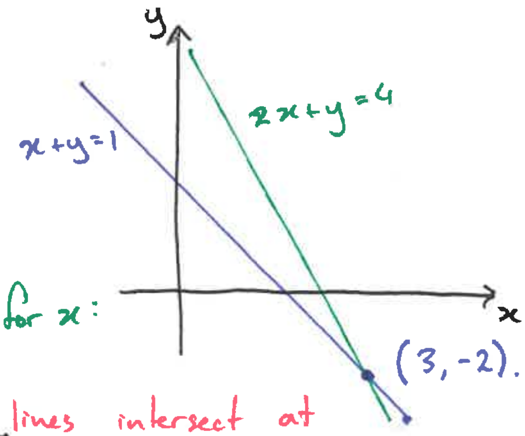
$$x + y = 1$$
$$-y = 2$$

• solve for y:

$$y = -2$$

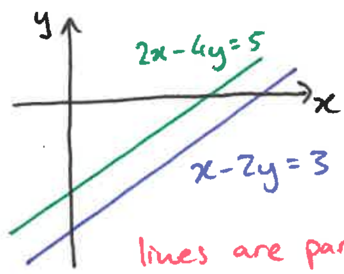
• sub. into eq. 1 and solve for x:

$$x - 2 = 1$$
$$x = 3.$$



lines intersect at a unique point.

(b) $$x - 2y = 3$$
$$2x - 4y = 5$$



lines are parallel!!!

• add $-2 \times$ eq 1 to eq 2:

$$x - 2y = 3$$
$$0x + 0y = -1$$

no solutions (the system is inconsistent)

(c) $$3x + y = 2$$
$$9x + 3y = 6$$

• add $-3 \times$ eq 1 to eq 2:

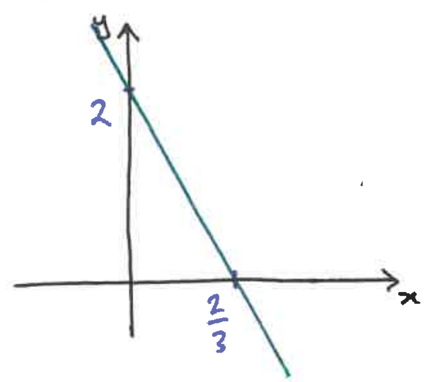
$$3x + y = 2$$
$$0 = 0$$

• use a parameter for y:

let $y = t$. Then $3x + t = 2$,

so $x = -\frac{1}{3}t + \frac{2}{3}$.

The solution is $x = -\frac{1}{3}t + \frac{2}{3}$, $y = t$.



the lines are the same!! (coincident)

The set of solutions of a linear equation in three variables is a plane, so a solution of a linear system in three variables corresponds to a point of intersection between planes.

Example 4. Solve the linear system and interpret the solution(s) geometrically.

$$\begin{aligned} x+y-z&=4 \\ 2x+2y-2z&=8 \\ 4x+4y-4z&=16 \end{aligned}$$

These three equations are equivalent, so the three planes are the same!!!

parametric soln: let ~~x~~ $y=s,\ z=t \Rightarrow x=-s+t+4$

Solution is the plane $x=-s+t+4,\ y=s,\ z=t$.

More generally, a linear system is usually solved by performing elementary row operations on the augmented matrix for the system.

$$\underbrace{\begin{aligned} x+y+2z&=9 \\ 2x+4y-3z&=1 \\ 3x+6y-5z&=0 \end{aligned}}_{\text{linear system}} \qquad \underbrace{\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}}_{\text{augmented matrix}}$$

$$\begin{aligned} 2x+0y-4z&=-2 \\ 0x+0y+z&=2 \\ 0x+y+0z&=1 \end{aligned} \qquad \begin{bmatrix} 2 & 0 & -4 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Elementary row operations.

1. Multiply a row by a nonzero constant.

eg. $$\begin{bmatrix} 2 & 0 & -4 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

2. Swap two rows.

eg. $$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

3. Add a multiple of one row to another row.

eg. $$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1+2R_3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Gauss-Jordan elimination

$\hookrightarrow$ $x=3,\ y=1,\ z=2$

Example 5. Solve the linear system and interpret the solution(s) geometrically.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\downarrow R_2 \to \tfrac{1}{2} R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - 3R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{bmatrix}$$

$$\downarrow R_3 \to -2R_3$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\downarrow R_1 \to R_1 - 2R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 + \tfrac{7}{2} R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\downarrow R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution is:

$x = 1, \; y = 2, \; z = 3.$

The three planes intersect at the point $(1,2,3)$.