Section 1.1: Introduction to Systems of Linear Equations

Objectives.

- Identify linear and nonlinear equations, and systems of linear equations.
- Understand terminology related to linear systems and matrices.
- Solve simple linear systems and interpret their solutions geometrically.
- Introduce elementary row operations.

A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where not all the a_i are zero.

A homogeneous linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, where not all the a_i are zero.

Example 1. Underline the linear equations. Circle the homogeneous linear equations.

$\underline{x + 4y = 9}$	$w + 3x - y^2 + z = 3$ not linear	$\underline{-3x + 2y - \frac{1}{2}z = 0}$
$x_1 - \sqrt{x_2} = 0$ not linear	$\underline{4x_1 - 2x_2 + 3x_3 = 0}$	$\underline{x_1 + x_2 + x_3 + x_4 = 1}$

A finite set of linear equations is called a system of linear equations (or linear system). The variables are called the unknowns.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

A solution of a linear system is an assignment of a number to each unknown so that each equation in the linear system is true.

Example 2. Decide whether each set of numbers is a solution to the linear system below.

$$\begin{aligned} x + y + 3z &= 0 \\ 2x + y - z &= 5 \end{aligned}$$

(a) $x = 0$, $y = 0$, $z = 0$

$0 + 0 + 3(0) = 0$

$2(0) + 0 - 0 = 0 \neq 5$

not a solution!!!

(b) $x = 5$, $y = -5$, $z = 0$

$5 + (-5) + 3(0) = 0$

$2(5) + (-5) - 0 = 5$

this is a solution!!

(c) $x = 1$, $y = 2$, $z = -1$

$1 + 2 + 3(-1) = 0$

$2(1) + 2 - (-1) = 5$

this is a solution!!

The set of solutions of a linear equation in x and y is a line in the xy-plane, so a solution of a linear system in x and y corresponds to a point of intersection between lines.

Example 3. Solve each linear system, and interpret the solution(s) geometrically.

(a) $$\begin{aligned} x + y &= 1 \\ 2x + y &= 4 \end{aligned}$$

• add $-2 \times$ eq 1 to eq 2:

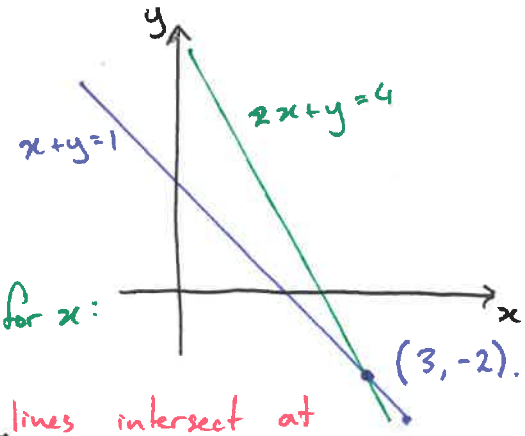
$$\begin{aligned} x + y &= 1 \\ -y &= 2 \end{aligned}$$

• solve for y:

$$y = -2$$

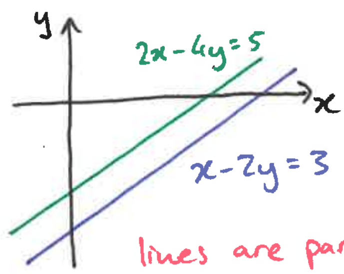
• sub. into eq. 1 and solve for x:

$$x - 2 = 1$$
$$x = 3.$$



lines intersect at a <u>unique</u> point.

(b) $$\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 5 \end{aligned}$$



lines are parallel!!!

• add $-2 \times$ eq 1 to eq 2:

$$\begin{aligned} x - 2y &= 3 \\ 0x + 0y &= -1 \end{aligned}$$

no solutions (the system is <u>inconsistent</u>)

(c) $$\begin{aligned} 3x + y &= 2 \\ 9x + 3y &= 6 \end{aligned}$$

• add $-3 \times$ eq 1 to eq 2:

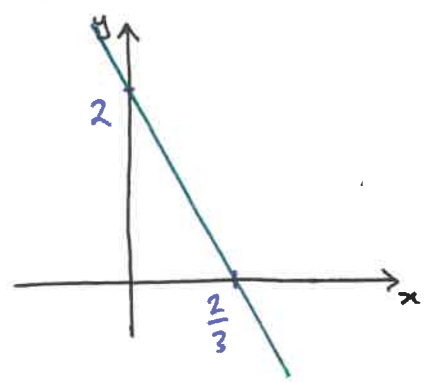
$$\begin{aligned} 3x + y &= 2 \\ 0 &= 0 \end{aligned}$$

• use a <u>parameter</u> for y:

let $y = t$. Then $3x + t = 2$,

so $x = -\frac{1}{3}t + \frac{2}{3}$.

The solution is $x = -\frac{1}{3}t + \frac{2}{3}$, $y = t$.



the lines are the same!! (coincident)

The set of solutions of a linear equation in three variables is a plane, so a solution of a linear system in three variables corresponds to a point of intersection between planes.

Example 4. Solve the linear system and interpret the solution(s) geometrically.

$$\begin{aligned} x + y - z &= 4 \\ 2x + 2y - 2z &= 8 \\ 4x + 4y - 4z &= 16 \end{aligned}$$

These three equations are equivalent, so the three planes are the same!!!

parametric sol$^{\underline{n}}$: let $y = s$, $z = t \Rightarrow x = -s + t + 4$

Solution is the plane $x = -s + t + 4$, $y = s$, $z = t$.

More generally, a linear system is usually solved by performing elementary row operations on the augmented matrix for the system.

$$\underbrace{\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}}_{\text{linear system}} \qquad \underbrace{\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}}_{\text{augmented matrix}}$$

$$\begin{aligned} 2x + 0y - 4z &= -2 \\ 0x + 0y + z &= 2 \\ 0x + y + 0z &= 1 \end{aligned} \qquad \begin{bmatrix} 2 & 0 & -4 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Elementary row operations.

1. Multiply a row by a nonzero constant.

eg. $$\begin{bmatrix} 2 & 0 & -4 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

2. Swap two rows.

eg. $$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

3. Add a multiple of one row to another row.

eg. $$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Gauss-Jordan elimination

$\hookrightarrow$ $x = 3$, $y = 1$, $z = 2$

Example 5. Solve the linear system and interpret the solution(s) geometrically.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\Big\downarrow \; R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\Big\downarrow \; R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\Big\downarrow \; R_2 \to \tfrac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\Big\downarrow \; R_3 \to R_3 - 3R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{bmatrix}$$

$$\Big\downarrow \; R_3 \to -2R_3$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Big\downarrow \; R_1 \to R_1 - 2R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Big\downarrow \; R_2 \to R_2 + \tfrac{7}{2}R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Big\downarrow \; R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution is:

$x = 1, \; y = 2, \; z = 3.$

The three planes intersect at the point $(1,2,3)$.

Section 1.2: Gaussian Elimination

Objectives.

- Identify matrices in row echelon form and reduced row echelon form.
- Use an augmented matrix in reduced row echelon form to write the solution for a linear system.
- Apply Gauss-Jordan elimination and Gaussian elimination to solve a linear system.
- Understand the relationship between numbers of unknowns, equations, and free variables.

A matrix is in row echelon form when the following are true.

(a) If a row contains a nonzero number, then the first nonzero number in the row is a 1. (This is a leading 1.)

(b) Any rows that contain only zeroes are at the bottom of the matrix.

(c) If a row has a leading 1, then it is further to the right than the leading 1 in any higher row.

A matrix is in reduced row echelon form if it is in row echelon form and:

(d) If a column contains a leading 1, then every other number in the column is 0.

Example 1. Which of the matrices below are in row echelon form (ref)? Which are in reduced row echelon form (rref)? Which are neither?

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$
r.e.f.

swap
$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -7 \end{bmatrix}$$
neither

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
r.r.e.f.

needs to be 1
$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$
neither

swap
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$
neither

$$\begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
r.r.e.f.

$$\begin{bmatrix} 1 & -4 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
r.e.f.

should be 1
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$
neither

swap
$$\begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$
neither

$$\begin{bmatrix} 1 & 2 & 4 & 0 & 8 \\ 1 & 0 & -5 & 2 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
neither

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$
r.r.e.f.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 7 \\ 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
r.r.e.f.

A variable corresponding to the leading 1 in some row is a leading variable. All other variables are free variables.

Example 2. Each augmented matrix below is in reduced row echelon form, and corresponds to a linear system in the variables x, y, and z. Find a solution for each linear system, identify the leading variables and the free variables, and describe the solution geometrically.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

leading 1s

leading vars: x, y, z free vars: none

Solution is $x=1$, $y=2$, $z=3$.

This is the point $(1,2,3)$.

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 4 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free variable

leading vars: x, y free vars: z

Let $z = t$. ← assign a parameter to each free var.

$x + 4t = -3 \Rightarrow x = -4t - 3$

$y + 2t = 0 \Rightarrow y = -2t$

The solution is $x = -4t-3$, $y = -2t$, $z = t$.

This is a line in three-dimensional space.

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

inconsistent system!!!

$0x + 0y + 0z = 1$.

leading vars: x, y free vars: z

System has no solution.

leading 1

(d) $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free variables

leading vars: x free vars: y, z

Let $y = s$, $z = t$.

Then $x - s - t = 0$, so $x = s + t$.

The sol$^{\underline{n}}$ is $x = s+t$, $y = s$, $z = t$.

This sol$^{\underline{n}}$ is a plane in three-dimensions.

Given an augmented matrix, an algorithm called <u>Gaussian elimination</u> can be used to find a matrix in row echelon form that has the same solutions.

<u>Gaussian elimination.</u>

1. Identify the leftmost column that contains a nonzero number.
2. If necessary, swap two rows so that the first number in this column is nonzero. Call this number a.
3. Multiply the top row by $\frac{1}{a}$ to create a leading 1.
4. Add multiples of the top row to each lower row so that every entry below the leading 1 is zero.
5. Cover the top row and repeat from Step 1.

Example 3. Apply Gaussian elimination to the augmented matrix below.

need a 1 at the top.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow \; R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow \; R_1 \to \tfrac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow \; R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\vdots$$

$$\downarrow \; R_2 \to -\tfrac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\downarrow \; R_3 \to R_3 - 5R_2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

$$\downarrow \; R_3 \to 2R_3$$

$$\underbrace{\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{\text{matrix is in row echelon form.}}$$

leading vars. are x_1, x_3, x_5.

free vars. are x_2, x_4.

While Gaussian elimination will result in a matrix in row echelon form, Gauss-Jordan elimination is an extension that gives a matrix in reduced row echelon form.

Gauss-Jordan elimination.

1. Perform Gaussian elimination to obtain a matrix in row echelon form.
2. Starting from the bottom row and working upwards, identify the leading 1 in each row (if there is one).
3. Add multiples of this row to each higher row so that each entry above the leading 1 is a zero.

Example 4. Solve the linear system.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 \phantom{{}-3x_6} &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \phantom{{}+4x_5} + 15x_6 &= 5 \\ 2x_1 + 6x_2 \phantom{{}-5x_3} + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$\downarrow$ $R_2 \to R_2 - 2R_1$, $R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$\downarrow$ $R_2 \to -R_2$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 6 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$\downarrow$ $R_3 \to R_3 - 5R_2$, $R_4 \to R_4 - 4R_2$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$\downarrow$ $R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$ $R_3 \to \frac{1}{6} R_3$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$ $R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$ $R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_2 = r$, $x_4 = s$, $x_5 = t$. Then:

$x_1 = -3r - 4s - 2t$

$x_3 = -2s$

$x_6 = \frac{1}{3}$.

A linear system is homogeneous if each of the equations in the system is homogeneous.

i.e.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

A homogeneous linear system in the variables $x_1, x_2, \ldots, x_n$ always has the trivial solution

$$x_1 = x_2 = \cdots = x_n = 0.$$

(Any solution where at least one variable is nonzero is called a nontrivial solution.)

Example 5. Solve the linear system. *Hint: compare this system with the previous example.*

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right]$$

elementary row operations do not affect a column of zeros!!

$\Big\downarrow$ row operations from Ex. 4.

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

r.r.e.f. from Ex. 4.

Let $x_2 = r$, $x_4 = s$, $x_5 = t$.

Then:

$x_1 = -3r - 4s - 2t$

$x_3 = -2s$

$x_6 = 0$

note: if $r = s = t = 0$, the solution is $(0,0,0,0,0,0)$.

trivial sol$\underline{\text{n}}$.

Theorem. A homogeneous linear system with n unknowns and r nonzero rows in the reduced row echelon form of the augmented matrix has $n - r$ free variables.

Theorem. A homogeneous linear system with more unknowns that equations has infinitely many solutions.

An alternative to Gauss-Jordan elimination is to use Gaussian elimination followed by back-substitution.

Gaussian elimination with back-substitution.

1. Perform Gaussian elimination to obtain a matrix in row echelon form.
2. Write an equation for each leading variable in terms of the other variables.
3. Starting from the bottom, substitute each equation into the equations above it.
4. Replace each free variable with a parameter.

Example 6. Use back-substitution to solve the linear system in Example 4.

From Ex. 4.

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ x_3 + 2x_4 + 3x_6 = 1 \\ x_6 = 1/3 \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2x_3 - 2x_5 \\ x_3 = 1 - 2x_4 - 3x_6 \\ x_6 = 1/3 \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2x_3 - 2x_5 \\ x_3 = 1 - 2x_4 - 3\left(\frac{1}{3}\right) = -2x_4 \\ x_6 = \frac{1}{3} \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2(-2x_4) - 2x_5 \\ \quad = -3x_2 - 4x_4 - 2x_5 \\ x_3 = -2x_4 \\ x_6 = 1/3 \end{cases}$$

The solution is:

$$x_1 = -3r - 4s - 2t$$
$$x_2 = r$$
$$x_3 = -2s$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = \tfrac{1}{3}.$$

Discussion. For each augmented matrix below, identify the number of solutions for the corresponding linear system.

$$\begin{bmatrix} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free variable, system is consistent $\Rightarrow$ infinite solutions

$$\begin{bmatrix} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

no free variable, consistent system $\Rightarrow$ one solution.

$$\begin{bmatrix} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$0 = 1$!!!!

no solutions (inconsistent system)

Section 1.3: Matrices and Matrix Operations

Objectives.

- Recognize a rectangular array of numbers as a matrix.
- Understand basic terminology and notation used for matrices.
- Apply the operations of matrix addition, subtraction, and multiplication correctly.
- Compute a linear combination of matrices.
- Find the transpose and the trace of a matrix.

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns. A square matrix of order n is a matrix with n rows and n columns.

eg. $\begin{bmatrix} 1 & 3 & 2 & 0 \\ 2 & 2 & 1 & 2 \\ 1 & -2 & 4 & 6 \end{bmatrix}$ is a 3×4 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a square matrix of order 2.

"main diagonal"

A matrix with one row is called a row vector (or row matrix). A matrix with one column is called a column vector (or column matrix).

row vector: $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ column vector: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Two matrices are equal if they have the same size and their corresponding entries are equal. If two matrices have the same size, then their sum (or difference) is found by adding (or subtracting) corresponding entries. A matrix can be multiplied by a scalar by multiplying each entry by the scalar.

Example 1. Simplify each expression.

(a) $$\begin{bmatrix} 3 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 \\ 4 & 2 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 3 & 2 & 1 \\ 1 & -5 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 & 7 \\ 1 & 2 & 3 & 0 \\ 5 & -3 & 9 & 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 7 & 3 & 0 & 2 \\ 5 & -1 & 2 & 1 \\ -2 & 2 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & -5 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & 1 \\ 5 & -2 & 0 & \cancel{-3} \\ -3 & 7 & -6 & -4 \end{bmatrix} -1$$

(c) $$2\begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -4 \\ 2 & -2 & 0 \\ -6 & 4 & 8 \end{bmatrix}$$

different sizes!!!

note: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is undefined!!!

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is an $m \times n$ matrix. The entry in the ith row and jth column of AB is found by multiplying each entry in the ith row of A by the corresponding entry in the jth column of B and adding the results.

$$\begin{array}{cc} & \\ & \\ & \\ i\text{th row} \rightarrow & \\ & \\ & \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{array}{c} \\ \end{array}\begin{array}{c} \begin{array}{ccccc} & & \downarrow j\text{th column} & & \end{array} \\ \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & \vdots & & \\ b_{r1} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \end{array} \qquad (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

Example 2. Compute each product below (if possible).

(a) $$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 \\ -2 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 6 \\ 9 & 17 & 7 \end{bmatrix}$$

$(1)(3) + (2)(-2) + (-1)(1) = -2$ $\qquad$ $(1)(4) + (2)(2) + (-1)(3) = 5$

$(2)(3) + (0)(-2) + (3)(1) = 9$ $\qquad$ $(2)(4) + (0)(2) + (3)(3) = 17$ $\qquad$ etc.

(b) $$\begin{bmatrix} 3 & 4 & -1 \\ -2 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix} = \text{undefined.}$$

$3 \times \underline{3}$ matrix $\qquad 2 \times 3$ matrix

dimensions do <u>not</u> match

(c) $$\begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 & 5 \\ 3 & 2 & 1 & -4 \\ -3 & 0 & 3 & 6 \end{bmatrix}$$

A matrix can be partitioned into submatrices by selecting certain rows and/or columns.

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$ where $A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$, $A_{21} = \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}$, $A_{22} = \begin{bmatrix} a_{33} \end{bmatrix}$

$$\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}$$ where $\vec{r}_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$, etc.

$$\left[\begin{array}{c|c|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{bmatrix}$$ where $\vec{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$, etc.

Partitioning matrices into rows and columns allows some different strategies for matrix multiplication. This is particularly useful when only some rows and/or columns of the product are needed.

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$$

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} B = \begin{bmatrix} \vec{a}_1 B \\ \vec{a}_2 B \\ \vdots \\ \vec{a}_n B \end{bmatrix}$$

Example 3. Simplify each expression.

(a) $$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

compare with last column of Ex. 2(a).

(b) $$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -3 \end{bmatrix} = 1\begin{bmatrix} -1 & 0 & 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & -4 \end{bmatrix}.$$

compare with 2nd row of Ex. 2(c).

If $A_1, A_2, \dots, A_n$ are matrices of the same size, and $c_1, c_2, \dots, c_n$ are scalars, then

$$c_1 A_1 + c_2 A_2 + \cdots + c_n A_n$$

is a linear combination of $A_1, A_2, \dots, A_n$.

When B is a column vector, the product AB is a linear combination of the columns of A.

Example 4. Simplify.

$$\begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y + z \\ -3x + 4y \\ -x + 8y + 5z \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} z$$

The last example suggests that we can express a linear system using matrix multiplication rather than an augmented matrix.

- linear system:

$$\begin{aligned} 2x + y + z &= 5 \\ -3x + 4y &= 2 \\ -x + 8y + 5z &= 0 \end{aligned}$$

equivalent ways of writing a linear system

- augmented matrix:

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ -3 & 4 & 0 & 2 \\ -1 & 8 & 5 & 0 \end{bmatrix}$$

- matrix equation:

$A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

If A is an $m \times n$ matrix, then the transpose A^T is the $n \times m$ matrix is obtained by swapping the rows and columns of A.

Example 5. Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 & 2 & 3 \\ -5 & 1 & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & -5 \\ 2 & 1 \\ 3 & 6 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

$B^T = \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$C^T = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}$

(d) $D = \begin{bmatrix} 3 & 4 & 0 \\ 4 & 5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

$D^T = \begin{bmatrix} 3 & 4 & 0 \\ 4 & 5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

note: $D^T = D$, so D is symmetric

Properties of transposes.

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$ note that the order is swapped!!!

The trace of a square matrix A, denoted by $\mathrm{tr}(A)$, is the sum of the entries on the main diagonal. (The trace is undefined for matrices that are not square.)

Example 6. Find the trace (if possible) of each matrix in the previous example.

tr A and tr B are undefined, because A and B are not square matrices.

$\mathrm{tr}(C) = 1 + 1 + 1 = 3$, $\mathrm{tr}(D) = 3 + 5 + (-1) = 7$.

Section 1.4: Inverses; Algebraic Properties of Matrices

Objectives.

- Learn the algebraic rules for matrix addition and multiplication.
- Understand zero matrices, identity matrices, and inverse matrices.
- Find the inverse of a 2×2 matrix.
- Use an inverse matrix to solve a linear system.
- Compute powers of matrices and matrix polynomials.

Many of the rules for matrix algebra will be familiar from previous mathematics classes.

Properties of matrix algebra. Lower case letters refer to scalars; upper case letters refer to matrices.

1. $A + B =$ $B + A$
2. $A + (B + C) =$ $(A + B) + C$
3. $A(BC) =$ $(AB)C$
4. $A(B \pm C) =$ $AB \pm AC$
5. $(A \pm B)C =$ $AC \pm BC$
6. $a(B \pm C) =$ $aB \pm aC$
7. $(a \pm b)C =$ $aC \pm bC$
8. $a(bC) =$ $(ab)C$
9. $a(BC) =$ $(aB)C = B(aC)$

Notice however that **matrix multiplication is <u>not</u> commutative**. That is, $AB \neq BA$ in general.

Example 1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. Compute AB and BA.

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

not equal!!!

$$BA = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{2\times 2}$$

The $m \times n$ matrix where every entry is 0 is a zero matrix and is denoted by $0_{m\times n}$.

e.g. $0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $0_{1\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

Properties of zero matrices.

1. $A \pm 0 =$ A
2. $A - A =$ 0
3. $\underset{\text{scalar}}{0}\underset{}{A} =$ $\underset{\text{matrix}}{0}$
4. If $cA = 0$, then either $c=0$ or $A=0$.

The last property listed above is called the zero-product principle. This is **not** true for matrix multiplication, as shown in the previous example.

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0_{2\times 2}$, but neither factor is zero.

It is also incorrect to 'cancel' factors in a matrix product.

Example 2. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 3 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 3 \\ -1 & -2 \end{bmatrix}$. Compute AB and AC.

$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

Thus $AB = AC$, but $B \neq C$.

A square matrix with 1 on the main diagonal and 0 everywhere else is called an identity matrix. This is denoted by either I or I_n (to specify the size of the matrix).

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.

Properties of identity matrices. Let A be an $m \times n$ matrix.

1. $AI_n =$ A
2. $I_m A =$ A

Example 3. Confirm the properties above for the matrix $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$.

$AI = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = A$

$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = A.$

If A is a square matrix, and B is a square matrix such that $AB = BA = I$, then we call A an invertible matrix (or nonsingular matrix) and we call B an inverse of A.

Example 4. Show that $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

$$AB = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$AB = I$ and $BA = I$, so $B = A^{-1}$.

If A does not have an inverse, then A is not invertible (or singular).

Example 5. Show that $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ is a singular matrix. guess: $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Suppose $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus:
$$\begin{aligned} 1a + 0c &= 1 \Rightarrow a = 1 \\ 1b + 0d &= 0 \Rightarrow b = 0 \\ 2a + 0c &= 0 \Rightarrow a = 0 \end{aligned}$$

contradiction!!!

We have found a contradiction, so A has no inverse.

Example 6. Show that if B and C are both inverses of A, then $B = C$.

By assumption, $AB = I$ and $CA = I$. Then:

$$B = IB = (CA)B = C(AB) = CI = C.$$

The previous example shows that if A is invertible then its inverse is unique. We denote this inverse by A^{-1}.

eg. $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$. (from Ex. 4).

Example 7. Show that if A and B are both invertible and have the same size, then $(AB)^{-1} = B^{-1}A^{-1}$.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Thus AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(The quantity $ad - bc$ is called the determinant of A. We study determinants in Chapter 2.)

Example 8. Decide whether each matrix is invertible, and find the inverse if possible.

(a) $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$$\det(A) = (6)(1) - (-3)(-2) = 6 - 6 = 0$$

A is not invertible. (ie. A is singular)

(b) $B = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}$

$$\det(B) = (5)(1) - (1)(3) = 5 - 3 = 2.$$

$\det B \neq 0$, so B has an inverse!!!

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}.$$

Recall that a linear system can be written in the form $A\vec{x} = \vec{b}$. If the coefficient matrix A is invertible, then the linear system can be solved by multiplying both sides of the matrix equation by A^{-1}.

Example 9. Solve the linear system.

$$5x + y = 2$$
$$3x + y = -2$$

$$\begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -8 \end{bmatrix}.$$

i.e. $x = 2$, $y = -8$.

if $A\vec{x} = \vec{b}$, then

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

A square matrix can be raised to any nonnegative integer power.

$$A^0 = I, \quad A^1 = A, \quad A^2 = AA, \quad A^3 = AAA, \ldots$$

An invertible matrix can be raised to any integer power (positive or negative).

$$A^{-n} = (A^{-1})^n.$$

Powers of invertible matrices. Let A be invertible, n be an integer, and k be a nonzero scalar.

1. A^{-1} is invertible, and $(A^{-1})^{-1} =$ A
2. A^n is invertible, and $(A^n)^{-1} =$ $(A^{-1})^n$
3. kA is invertible, and $(kA)^{-1} =$ $k^{-1}A^{-1} = \frac{1}{k}A^{-1}.$

If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial and A is a square matrix, then

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

Example 10. Let $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ and let $p(x) = x^2 - x + 3$.

(a) Compute A^3.

$$A^3 = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}.$$

(b) Compute $p(A)$.

$$p(A) = A^2 - A + 3I = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 8 & 3 \end{bmatrix}.$$

Recall that the transpose of a matrix is found by swapping the rows and the columns of the matrix.

Example 11. Show that if A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

$$A^T(A^{-1})^T = (AA^{-1})^T = I^T = I.$$

$$(A^{-1})^T A^T = (A^{-1}A)^T = I^T = I.$$

Therefore, $(A^T)^{-1} = (A^{-1})^T.$

Section 1.5: Elementary Matrices and a Method for Finding A^{-1}

Objectives.

- Write each elementary row operation using matrix multiplication.
- Find the inverse of a given row operation.
- Use row operations to find the inverse of a matrix or show that the matrix is not invertible.

Recall the three elementary row operations:

- multiply one row by a constant
- swap two rows
- add a multiple of one row to another row.

Two matrices A and B are row equivalent if A can be transformed into B using elementary row operations.

eg. $\begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are row equivalent.

(i.e) $R_1 \leftrightarrow R_2$, then $R_1 \to \frac{1}{4} R_1$

An elementary matrix is a matrix that can be obtained from an identity matrix using a single elementary row operation. Multiplication by an elementary matrix is the same as performing an elementary row operation.

Example 1. What elementary row operation is equivalent to calculating EB for each matrix E below?

(a) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← double row 2 of B.

$R_2 \longrightarrow 2R_2$

(c) $E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Swap row 1 and row 3.

$R_1 \leftrightarrow R_3$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← add -3 times row 1 to row 2

$R_2 \to R_2 - 3R_1$

eg. $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

(d) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← multiply row 1 by 1.

$R_1 \to R_1$

Each elementary row operation can be reversed by applying another elementary row operation.

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 5R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2. Find an elementary 3×3 matrix that corresponds to each row operation, and find an elementary row operation that reverses each row operation.

	elementary matrix	inverse row operation
(a) multiply row 3 by $-\frac{1}{5}$ i.e. $R_3 \to -\frac{1}{5}R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{5} \end{bmatrix}$	$R_3 \to -5R_3$ i.e. multiply by reciprocal.
(b) swap row 1 and row 2 i.e. $R_1 \leftrightarrow R_2$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$ i.e. swap rows again!!!
(c) add 4 times row 2 to row 1 i.e. $R_1 \to R_1 + 4R_2$	$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \to R_1 - 4R_2$ i.e. ~~subtract~~ subtract $4R_2$ from R_1.

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution. ← note: $A\vec{x} = \vec{0}$ always has the solution $\vec{x} = \vec{0}$. If A is invertible, then $\vec{x} = \vec{0}$ is the only solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.

The Equivalence Theorem says that if A is invertible then there is a sequence of elementary row operations that reduces A to I_n. The same sequence of row operations applied to I_n results in the matrix A^{-1}.

> **Inverting a matrix.** To find the inverse of an $n \times n$ matrix A:
>
> 1. Form the matrix $[A|I_n]$.
> 2. Apply elementary row operations to reduce A to I_n.
> 3. The resulting matrix has the form $[I_n|A^{-1}]$.

Example 3. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

• start with $[A \mid I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \quad \begin{aligned} R_2 &\to R_2 - 2R_1 \\ R_3 &\to R_3 - R_1 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \quad R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\Big\downarrow \quad R_3 \to -R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\Big\downarrow \quad \begin{aligned} R_2 &\to R_2 + 3R_3 \\ R_1 &\to R_1 - 3R_3 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\Big\downarrow \quad R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

... finish with $[I \mid A^{-1}]$.

Therefore:

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

The algorithm for finding an inverse matrix can also be used to decide whether a matrix has an inverse.

Example 4. Determine whether $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 3 & 5 & 11 \end{bmatrix}$ is invertible and find the inverse if possible.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 5 & 11 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 10 & -2 & 1 & 0 \\ 0 & 2 & 20 & -3 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 - 2R_2$$

uh-oh!!! $\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 10 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$

A is not invertible, because we cannot reduce A to I_3 using elementary row operations.

Example 5. Decide whether each homogeneous linear system has nontrivial solutions.

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad + 8x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is invertible (Ex. 3), so the system has <u>only</u> the trivial solution.

(b)
$$\begin{aligned} x_1 + x_2 - 3x_3 &= 0 \\ 2x_1 + 3x_2 + 4x_3 &= 0 \\ 3x_1 + 5x_2 + 11x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 3 & 5 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The coefficient matrix is <u>not</u> invertible (Ex. 4), so there are nontrivial solutions.

Section 1.6: More on Linear Systems and Invertible Matrices

Objectives.

- Use an inverse matrix to solve a linear system.
- Understand properties of invertible matrices.
- Determine all vectors $\vec{b}$ for which the linear system $A\vec{x} = \vec{b}$ is consistent.

Theorem. A linear system has either no solutions, exactly one solution, or an infinite number of solutions.

Proof. Suppose $\vec{x}_1$ and $\vec{x}_2$ are distinct solutions of $A\vec{x} = \vec{b}$.
Let $\vec{x}_0 = \vec{x}_1 - \vec{x}_2$. Then $\vec{x}_0 \neq \vec{0}$ because $\vec{x}_1 \neq \vec{x}_2$. Also:

$$A\vec{x}_0 = A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}.$$

If k is any scalar, then:

$$A(\vec{x}_1 + k\vec{x}_0) = A\vec{x}_1 + kA\vec{x}_0 = \vec{b} + k\cdot\vec{0} = \vec{b} + \vec{0} = \vec{b}.$$

That is, $\vec{x}_1 + k\vec{x}_0$ is a sol$^{\text{n}}$ of $A\vec{x} = \vec{b}$ for any k.
Therefore, this system has infinitely many solutions.

Theorem. If A is an invertible $n \times n$ matrix, and $\vec{b}$ is an $n \times 1$ column vector, then the linear system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

From the previous theorem, if A is invertible then the system $A\vec{x} = \vec{b}$ can be solved by multiplying by A^{-1}.

Example 1. Solve the linear system.

$$\begin{aligned} 6x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + x_3 &= 0 \\ 10x_1 + 3x_2 + 4x_3 &= -1 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

The inverse of $A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$.

Thus:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ 4 \end{bmatrix}.$$

Sometimes we may want to solve several linear systems that have the same coefficient matrix A. For instance, supose that we want to solve all of the systems:

$$A\vec{x} = \vec{b}_1, \quad A\vec{x} = \vec{b}_2, \quad \dots, \quad A\vec{x}_k = \vec{b}_k.$$

If A is invertible, then the solutions can be found using matrix multiplication.

$$\vec{x}_1 = A^{-1}\vec{b}_1, \quad \vec{x}_2 = A^{-1}\vec{b}_2, \quad \dots, \quad \vec{x}_k = A^{-1}\vec{b}_k$$

An alternate approach (which also works when A is singular!) is to solve the systems at the same time by row-reducing the augmented matrix

$$\left[A \,|\, \vec{b}_1 \,|\, \vec{b}_2 \,|\, \cdots \,|\, \vec{b}_k \right] \xrightarrow{\text{row operations}} \left[I \,|\, \vec{x}_1 \,|\, \vec{x}_2 \,|\, \cdots \,|\, \vec{x}_k \right].$$

Example 2. Solve the linear systems.

(a)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ x_2 - 2x_3 &= -2 \\ 2x_1 - 3x_2 + 2x_3 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 1 \\ x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 + 2x_3 &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 2 & -3 & 2 & 4 & -1 \end{array}\right]$$

$$\big\downarrow \; R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 3 & -6 & -6 & -3 \end{array}\right]$$

$$\big\downarrow \; R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

$$\big\downarrow \; R_1 \to R_1 + 3R_2$$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & -2 & -1 & 4 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

part (a): $x_3 = t$, so
$x_1 = -1 + 2t$, $x_2 = -2 + 2t$.

part (b): system is inconsistent, so there are no solutions.

Our *definition* of an inverse matrix $B = A^{-1}$ requires that both $AB = I$ and $BA = I$ are true. However, it is enough to know that at least one of these equations is true.

Theorem. Let A and B be a square matrices. If $AB = I$ or $BA = I$, then $B = A^{-1}$.

Example 3. Show that $B = A^{-1}$ for the matrices A and B below. (These are the matrices from Example 1.)

$$A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

From the Thm above, we only need to show $AB = I$ (or $BA = I$).

$$AB = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Therefore, $B = A^{-1}$. (also, $A = B^{-1}$).

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.

from Sect. 1.5 (page 2)

5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.

new conditions.

Theorem. If A and B are square matrices and AB is invertible, then both A and B are invertible.

Proof. Suppose $\vec{x}_0$ is a sol'n to $B\vec{x} = \vec{0}$. Then

$$(AB)\vec{x}_0 = A(B\vec{x}_0) = A\vec{0} = \vec{0}.$$

Because AB is invertible, the system $(AB)\vec{x} = \vec{0}$ has only the trivial solution. Thus $\vec{x}_0 = \vec{0}$. That is, $B\vec{x} = \vec{0}$ has only the trivial solution, so B is invertible.

Thus $A = A(BB^{-1}) = \underbrace{(AB)B^{-1}}_{\text{product of invertible matrices is invertible.}}$ is invertible.

Problem. Given an $m \times n$ matrix A, find all $m \times 1$ vectors $\vec{b}$ for which the linear system $A\vec{x} = \vec{b}$ is consistent.

If A is invertible, this problem is easy. ($A\vec{x} = \vec{b}$ is consistent for every $m \times 1$ vector $\vec{b}$.) Otherwise, row operations can be used to determine which vectors $\vec{b}$ give consistent systems.

Example 4. What conditions must b_1, b_2, b_3 satisfy for the system below to be consistent?

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= b_1 \\ x_2 - 2x_3 &= b_2 \\ 2x_1 - 3x_2 + 2x_3 &= b_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 2 & -3 & 2 & b_3 \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 3 & -6 & b_3 - 2b_1 \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 - 3b_2 \end{array}\right]$$

For this system to be consistent, we need

$b_3 - 2b_1 - 3b_2 = 0$, so $b_3 = 2b_1 + 3b_2$.

Section 1.7: Diagonal, Triangular, and Symmetric Matrices

Objectives.

- Identify diagonal, upper triangular, lower triangular, and symmetric matrices.
- Understand properties of diagonal, triangular, and symmetric matrices.

Some matrices are easier to compute with than others, either because they contain a lot of zeroes or because or their symmetry. These matrices will be important in some of the topics we study later in this course.

A square matrix A is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- diagonal if the only nonzero entries are on the main diagonal.

  i.e. $a_{ij} = 0$ if $i \neq j$

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

- upper triangular if every entry below the main diagonal is zero.

  i.e. $a_{ij} = 0$ if $i > j$

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

- lower triangular if every entry above the main diagonal is zero.

  i.e. $a_{ij} = 0$ if $i < j$

- symmetric if $A = A^T$.

  i.e. $a_{ij} = a_{ji}$

Example 1. Identify each matrix as diagonal and/or upper triangular and/or lower triangular and/or symmetric.

$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$

diagonal, upper Δ, lower Δ, symm.

$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$

symmetric

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ -3 & 8 & 3 \end{bmatrix}$

lower Δ

$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

upper Δ

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

none of these!! (not square).

$\begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

upper Δ

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$

symmetric

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

diagonal, upper Δ, lower Δ, symm.

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

diagonal, upper Δ, lower Δ, symmetric

An $n \times n$ diagonal matrix can be written in the form

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

This matrix is invertible if and only if every entry on the main diagonal is nonzero, in which case the inverse is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$, provided none of the d_i values is zero.

Example 2. Compute each inverse (if it exists!).

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{-1} =$ DNE!!!

If k is a positive integer, then D^k can be computed by raising each (nonzero) entry in D to the power k.

Example 3. Simplify each expression (if possible!).

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{3} = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & (\frac{1}{3})^3 & 0 \\ 0 & 0 & 1^3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & \frac{1}{27} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-4} = \begin{bmatrix} \frac{1}{16} & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{-4} =$ DNE!!!

cannot find D^{-1}, so we also cannot find D^{-4}.

Multiplication by a diagonal matrix is also relatively simple.

Example 4. Compute each product.

(a) $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 10 \\ 3 & 4 \end{bmatrix}$

i.e. multiply R_1 by -3
multiply R_2 by 2
multiply R_3 by 1

(b) $\begin{bmatrix} 4 & -1 & 2 \\ 8 & 1 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 4 \\ 8 & 4 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

i.e. multiply C_1 by 1
" C_2 by 4
" C_3 by 2

Properties of (upper) triangular matrices. *Note: similar properties hold for lower triangular matrices.*

1. The transpose of an upper triangular matrix is lower triangular.
2. The product of two upper triangular matrices is upper triangular.
3. An upper triangular matrix is invertible if and only if every entry on the main diagonal is nonzero.
4. The inverse of an invertible upper triangular matrix is upper triangular.

Example 5. Suppose that $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Show that $A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } A^{-1} = \begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}.$$

(b) Compute AB and BA. What do you notice?

$$AB = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

Proof of 2. Suppose A, B are upper triangular, and let $C = AB$. If $i > j$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$= \underbrace{a_{i1}b_{1j} + \cdots + a_{i(i-1)}b_{(i-1)j}}_{a_{i1}=0,\ \dots,\ a_{i(i-1)}=0} + \underbrace{a_{ii}b_{ij} + \cdots + a_{in}b_{nj}}_{b_{ij}=0,\ \dots,\ b_{nj}=0}$$

$$= \underline{0}.$$

Because $c_{ij} = 0$ when $i > j$, the matrix C is upper triangular.

A symmetric $\iff$ $A = A^T$

Properties of symmetric matrices. If A and B are symmetric $n \times n$ matrices, and k is a scalar, then:

1. A^T is symmetric.
2. $A + B$ and $A - B$ are both symmetric.
3. kA is symmetric
4. AB is symmetric if and only if $AB = BA$.
5. If A is invertible then A^{-1} is symmetric.
6. If A is invertible, then AA^T and A^TA are invertible.

Proof of 2.

$$(A+B)_{ij} = A_{ij} + B_{ij} = A_{ji} + B_{ji} = (A+B)_{ji}.$$

row i, col. j of $A+B$

The next example illustrates property 4 above.

Example 6. Compute each product.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ } AB is symmetric, so $AB = BA$.

(d) $\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

One final observation is that for <u>any</u> matrix A, the products AA^T and A^TA are both symmetric.

$$(AA^T)^T = (A^T)^T A^T = AA^T, \text{ so } AA^T \text{ is symmetric.}$$

Swap order when using transpose!!!

Example 7. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix}$. Confirm that both AA^T and A^TA are symmetric.

$$AA^T = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 19 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 3 & 7 \\ 3 & 1 & 3 \\ 7 & 3 & 10 \end{bmatrix}.$$

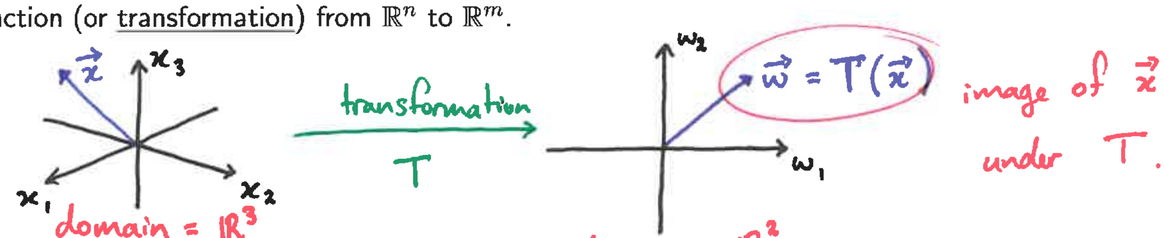
Section 1.8: Introduction to Linear Transformations

Objectives.

- Understand an $m \times n$ matrix as a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Identity the standard basis vectors for $\mathbb{R}^n$ and the standard matrix of a transformation.
- Study some simple linear transformations.

The set of all $n \times 1$ column vectors is denoted by $\mathbb{R}^n$. In this section, we interpret multiplication by an $m \times n$ matrix as a function (or transformation) from $\mathbb{R}^n$ to $\mathbb{R}^m$.

eg.



Example 1. The set of linear equations

$$\begin{aligned} w_1 &= x_1 - 2x_2 + 4x_3 - 2x_4 \\ w_2 &= 3x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= -6x_1 \phantom{{}+ x_2} + x_3 - x_4 \end{aligned}$$

defines a linear transformation T_A from $\mathbb{R}^4$ to $\mathbb{R}^3$.

(a) Express the transformation T_A using matrix multiplication.

$$T_A(\vec{x}) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

or: $\vec{w} = T_A(\vec{x}) = A\vec{x}$

where $A = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix}$.

(b) Find the image of the vector $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ under the transformation T_A.

$$T_A\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ -4 \end{bmatrix}$$

Note: The linear transformation in this example can also be written in comma-delimited form as

$$T(x_1, x_2, x_3, x_4) = (\underbrace{x_1 - 2x_2 + 4x_3 - 2x_4}_{w_1}, \underbrace{3x_1 + x_2 - 2x_3 + x_4}_{w_2}, \underbrace{-6x_1 + x_3 - x_4}_{w_3}).$$

Two simple matrix transformations are the zero transformation/operator and the identity transformation/operator.

$T_0(\vec{x}) = 0\vec{x} = \vec{0}$ "zero transformation"

$T_I(\vec{x}) = I\vec{x} = \vec{x}$ "identity transformation"

Properties of matrix transformations. If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and k is a scalar, then:

1. $T_A(\vec{0}) = \vec{0}$ $\longrightarrow$ the zero vector/origin is unchanged by a matrix transformation
2. $T_A(k\vec{u}) = k\,T_A(\vec{u})$ $\longrightarrow$ "homogeneity"
3. $T_A(\vec{u}+\vec{v}) = T_A(\vec{u}) + T_A(\vec{v})$ $\longrightarrow$ "additive property"

Not all transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are matrix transformations. For instance:

$$w_1 = x_1 + x_2^2$$
$$w_2 = x_1 x_2$$

is not a matrix transformation.

"non linear terms"

However, a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies both homogeneity and the additivity property is a matrix transformation.

(More specifically, if these two properties are satisfied then T is called a linear transformation. That is, every matrix transformation is a linear transformation, and every linear transformation is a matrix transformation.)

Example 2. Show that $T(x,y) = (x+3y, 2x, 2x-y)$ is a linear transformation.

Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$. Then:

$$T(k\vec{u}) = T(ku_1, ku_2) = (ku_1 + 3ku_2, 2ku_1, 2ku_1 - ku_2)$$
$$= k(u_1 + 3u_2, 2u_1, 2u_1 - u_2) = k\,T(u_1,u_2) = k\,T(\vec{u}).$$

$$T(\vec{u}+\vec{v}) = T(u_1+v_1, u_2+v_2) = (u_1+v_1+3(u_2+v_2), 2(u_1+v_1), 2(u_1+v_1)-(u_2+v_2))$$
$$= (u_1+3u_2, 2u_1, 2u_1-u_2) + (v_1+3v_2, 2v_1, 2v_1-v_2)$$
$$= T(\vec{u}) + T(\vec{v}).$$

T satisfies homogeneity and the additive property, so T is a linear transformation

Theorem. If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and $T_A(\vec{x}) = T_B(\vec{x})$ for every vector $\vec{x}$ in $\mathbb{R}^n$, then $A = B$.

As a consequence of this theorem, each linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ corresponds to exactly one $m \times n$ matrix, which we call the standard matrix for the transformation.

eg. the standard matrix in Ex. 1 is $A = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix}$.

The standard basis vectors for $\mathbb{R}^n$ are the $n \times 1$ vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Every vector in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

eg. in $\mathbb{R}^3$: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$.

Example 3. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$

or: $T(x_1, x_2) = (x_1 - x_2,\ 2x_1 + x_2,\ x_1 + 3x_2)$.

(a) Compute $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2-3 \\ 2(2)+3 \\ 2+3(3) \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 11 \end{bmatrix}.$$

"image of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under T"

(b) Find the image of each standard basis vector in $\mathbb{R}^2$.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-0 \\ 2(1)+0 \\ 1+3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-1 \\ 2(0)+1 \\ 0+3(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

(c) Find the standard matrix for this linear transformation.

$$A = \left[\, T(\vec{e}_1) \mid T(\vec{e}_2) \,\right] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Example 4. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}-1\\-2\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = \begin{bmatrix}10\\0\end{bmatrix}.$$

(a) Find the standard matrix for T.

- write $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$.

$$\begin{bmatrix}1\\0\end{bmatrix} = a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}2\\2\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix} = c\begin{bmatrix}1\\-1\end{bmatrix} + d\begin{bmatrix}2\\2\end{bmatrix}$$

- solve for constants.

$$a = \tfrac{1}{2}, \quad b = \tfrac{1}{4} \qquad\qquad c = -\tfrac{1}{2}, \quad d = \tfrac{1}{4}.$$

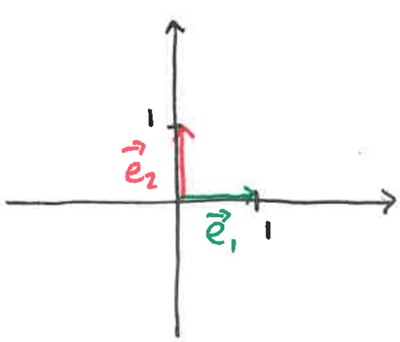
- find $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

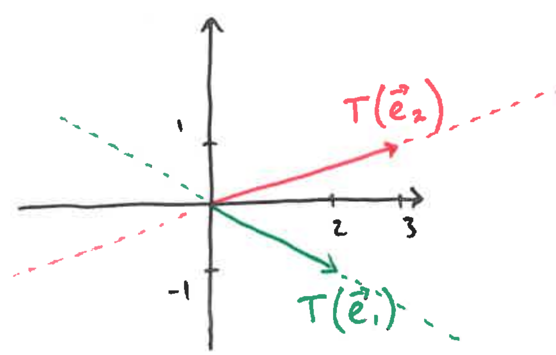
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\tfrac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}2\\2\end{bmatrix}\right) = \tfrac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \tfrac{1}{4}T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix}-1\\-2\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}10\\0\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = -\tfrac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \tfrac{1}{4}T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = -\tfrac{1}{2}\begin{bmatrix}-1\\-2\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}10\\0\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}.$$

$$A = \left[\,T(\vec{e}_1) \,\middle|\, T(\vec{e}_2)\,\right] = \begin{bmatrix}2 & 3\\-1 & 1\end{bmatrix}.$$

(b) Sketch a diagram showing each standard basis vector in $\mathbb{R}^2$, and another showing the image of each standard basis vector under the transformation T.

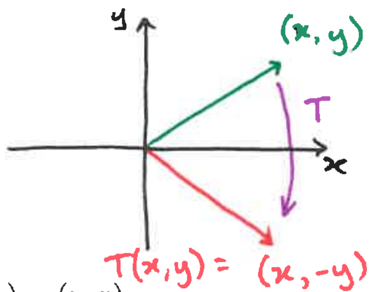




A linear transformation can be interpreted geometrically as a distortion of space that preserves straight lines. (The origin should also remain unchanged!) Some simple examples of these transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ include reflections, (orthogonal) projections, and rotations.

Example 5. For each transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, sketch a diagram showing a typical vector $\vec{x}$ and its image $T(\vec{x})$. Then describe the transformation and find the standard matrix for the transformation.

(a) $T(x, y) = (x, -y)$

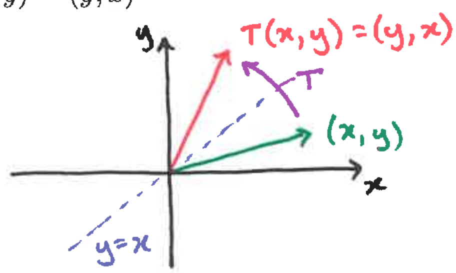


- reflection in x-axis.

$T(1,0) = (1,0)$, $T(0,1) = (0,-1)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) $T(x, y) = (y, x)$

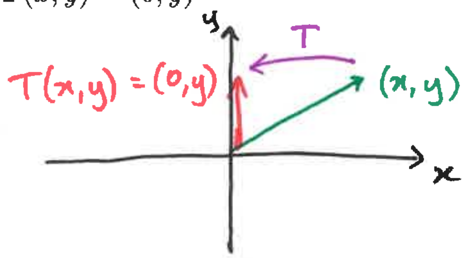


- reflection in the line $y = x$.

$T(1,0) = (0,1)$, $T(0,1) = (1,0)$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) $T(x, y) = (0, y)$



- (orthogonal) projection onto y-axis

$T(1,0) = (0,0)$, $T(0,1) = (0,1)$.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 6. Describe each transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ and find the standard matrix for the transformation.

(a) $T(x, y, z) = (x, 0, z)$

- orthogonal projection onto xz-plane

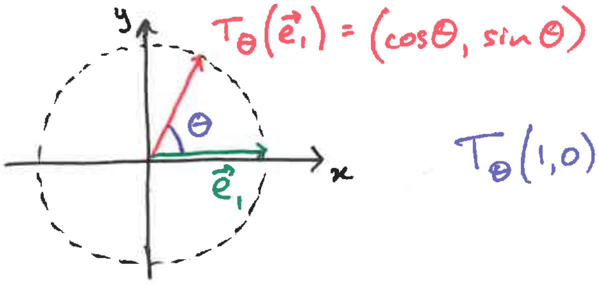
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $T(x, y, z) = (x, y, -z)$

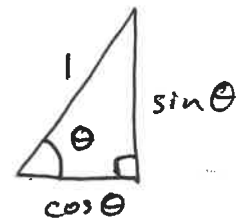
- reflection in xy-plane

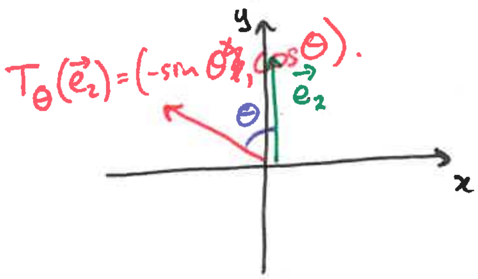
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In $\mathbb{R}^2$, rotation about the origin by an angle θ is a linear transformation. We can find the standard matrix for this rotation by considering the image of the standard basis vectors.



$$T_\theta(1,0) = (\cos\theta, \sin\theta)$$





$$T_\theta(0,1) = (-\sin\theta, \cos\theta).$$

$$\underbrace{R_\theta}_{\text{counterclockwise rotation by } \theta.} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example 7. Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represents a rotation of $45°$ about the origin.

(a) Find the standard matrix for the transformation.

$$R_\theta = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

(b) Find the image of $\vec{x} = (1, 4)$ under this transformation.

$$T_\theta(1,4) = R_\theta \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} -2.12 \\ 3.54 \end{bmatrix}.$$