Section 1.8: Introduction to Linear Transformations

Objectives.

- Understand an $m \times n$ matrix as a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Identity the standard basis vectors for $\mathbb{R}^n$ and the standard matrix of a transformation.
- Study some simple linear transformations.

The set of all $n \times 1$ column vectors is denoted by $\mathbb{R}^n$. In this section, we interpret multiplication by an $m \times n$ matrix as a function (or transformation) from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Example 1. The set of linear equations

$$\begin{aligned} w_1 &= x_1 - 2x_2 + 4x_3 - 2x_4 \\ w_2 &= 3x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= -6x_1 \phantom{{}+2x_2} + x_3 - x_4 \end{aligned}$$

defines a linear transformation T_A from $\mathbb{R}^4$ to $\mathbb{R}^3$.

(a) Express the transformation T_A using matrix multiplication.

(b) Find the image of the vector $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ under the transformation T_A.

Note: The linear transformation in this example can also be written in comma-delimited form as

$$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + 4x_3 - 2x_4, 3x_1 + x_2 - 2x_3 + x_4, -6x_1 + x_3 - x_4).$$

Two simple matrix transformations are the zero transformation/operator and the identity transformation/operator.

Properties of matrix transformations. If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and k is a scalar, then:

1. $T_A(\vec{0}) = \vec{0}$
2. $T_A(k\vec{u}) = k\,T_A(\vec{u})$
3. $T_A(\vec{u} + \vec{v}) = T_A(\vec{u}) + T_A(\vec{v})$

Not all transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are matrix transformations. For instance:

However, a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies both homogeneity and the additivity property is a matrix transformation.

(More specifically, if these two properties are satisfied then T is called a linear transformation. That is, every matrix transformation is a linear transformation, and every linear transformation is a matrix transformation.)

Example 2. Show that $T(x, y) = (x + 3y, 2x, 2x - y)$ is a linear transformation.

Theorem. If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and $T_A(\vec{x}) = T_B(\vec{x})$ for every vector $\vec{x}$ in $\mathbb{R}^n$, then $A = B$.

As a consequence of this theorem, each linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ corresponds to exactly one $m \times n$ matrix, which we call the standard matrix for the transformation.

The standard basis vectors for $\mathbb{R}^n$ are the $n \times 1$ vectors

Every vector in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

Example 3. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

(b) Find the image of each standard basis vector in $\mathbb{R}^2$.

(c) Find the standard matrix for this linear transformation.

Example 4. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \qquad \text{and} \qquad T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

(a) Find the standard matrix for T.

(b) Sketch a diagram showing each standard basis vector in $\mathbb{R}^2$, and another showing the image of each standard basis vector under the transformation T.

A linear transformation can be interpreted geometrically as a distortion of space that preserves straight lines. (The origin should also remain unchanged!) Some simple examples of these transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ include reflections, (orthogonal) projections, and rotations.

Example 5. For each transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, sketch a diagram showing a typical vector $\vec{x}$ and its image $T(\vec{x})$. Then describe the transformation and find the standard matrix for the transformation.

(a) $T(x, y) = (x, -y)$

(b) $T(x, y) = (y, x)$

(c) $T(x, y) = (0, y)$

Example 6. Describe each transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ and find the standard matrix for the transformation.

(a) $T(x, y, z) = (x, 0, z)$

(b) $T(x, y, z) = (x, y, -z)$

In $\mathbb{R}^2$, rotation about the origin by an angle θ is a linear transformation. We can find the standard matrix for this rotation by considering the image of the standard basis vectors.

Example 7. Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represents a rotation of 45° about the origin.

(a) Find the standard matrix for the transformation.

(b) Find the image of $\vec{x} = (1, 4)$ under this transformation.