Section 1.6: More on Linear Systems and Invertible Matrices

Objectives.

- Use an inverse matrix to solve a linear system.
- Understand properties of invertible matrices.
- Determine all vectors $\vec{b}$ for which the linear system $A\vec{x} = \vec{b}$ is consistent.

Theorem. A linear system has either no solutions, exactly one solution, or an infinite number of solutions.

Proof.

Theorem. If A is an invertible $n \times n$ matrix, and $\vec{b}$ is an $n \times 1$ column vector, then the linear system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

From the previous theorem, if A is invertible then the system $A\vec{x} = \vec{b}$ can be solved by multiplying by A^{-1}.

Example 1. Solve the linear system.

$$\begin{aligned} 6x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + x_3 &= 0 \\ 10x_1 + 3x_2 + 4x_3 &= -1 \end{aligned}$$

Sometimes we may want to solve several linear systems that have the same coefficient matrix A. For instance, supose that we want to solve all of the systems:

If A is invertible, then the solutions can be found using matrix multiplication.

An alternate approach (which also works when A is singular!) is to solve the systems at the same time by row-reducing the augmented matrix

Example 2. Solve the linear systems.

(a)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ x_2 - 2x_3 &= -2 \\ 2x_1 - 3x_2 + 2x_3 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 1 \\ x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 + 2x_3 &= -1 \end{aligned}$$

Our *definition* of an inverse matrix $B = A^{-1}$ requires that both $AB = I$ and $BA = I$ are true. However, it is enough to know that at least one of these equations is true.

Theorem. Let A and B be a square matrices. If $AB = I$ or $BA = I$, then $B = A^{-1}$.

Example 3. Show that $B = A^{-1}$ for the matrices A and B below. (These are the matrices from Example 1.)

$$A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

Equivalence Theorem. If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A can be written as a product of elementary matrices.
5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.

Theorem. If A and B are square matrices and AB is invertible, then both A and B are invertible.

Proof.

Problem. Given an $m \times n$ matrix A, find all $m \times 1$ vectors $\vec{b}$ for which the linear system $A\vec{x} = \vec{b}$ is consistent.

If A is invertible, this problem is easy. ($A\vec{x} = \vec{b}$ is consistent for every $m \times 1$ vector $\vec{b}$.) Otherwise, row operations can be used to determine which vectors $\vec{b}$ give consistent systems.

Example 4. What conditions must b_1, b_2, b_3 satisfy for the system below to be consistent?

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= b_1 \\ x_2 - 2x_3 &= b_2 \\ 2x_1 - 3x_2 + 2x_3 &= b_3 \end{aligned}$$