## Section 1.1: Introduction to Systems of Linear Equations

**Objectives.**

- Identify linear and nonlinear equations, and systems of linear equations.
- Understand terminology related to linear systems and matrices.
- Solve simple linear systems and interpret their solutions geometrically.
- Introduce elementary row operations.

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A linear equation in the variables $x_1, x_2, \dots, x_n$ is an equation of the form

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where not all the $a_i$ are zero.

A homogeneous linear equation in the variables $x_1, x_2, \dots, x_n$ is an equation of the form

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, where not all the $a_i$ are zero.

**Example 1.** Underline the linear equations. Circle the homogeneous linear equations.

| | | |
|---|---|---|
| $x + 4y = 9$ | $w + 3x - y^2 + z = 3$ not linear | $-3x + 2y - \frac{1}{2}z = 0$ |
| $x_1 - \sqrt{x_2} = 0$ not linear | $4x_1 - 2x_2 + 3x_3 = 0$ | $x_1 + x_2 + x_3 + x_4 = 1$ |

A finite set of linear equations is called a system of linear equations (or linear system). The variables are called the unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \qquad \vdots \qquad \qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A solution of a linear system is an assignment of a number to each unknown so that each equation in the linear system is true.

**Example 2.** Decide whether each set of numbers is a solution to the linear system below.

$$\begin{aligned} x + y + 3z &= 0 \\ 2x + y - z &= 5 \end{aligned}$$

(a) $x = 0$, $y = 0$, $z = 0$

$0 + 0 + 3(0) = 0$

$2(0) + 0 - 0 = 0 \neq 5$

not a solution!!!

(b) $x = 5$, $y = -5$, $z = 0$

$5 + (-5) + 3(0) = 0$

$2(5) + (-5) - 0 = 5$

this is a solution!!

(c) $x = 1$, $y = 2$, $z = -1$

$1 + 2 + 3(-1) = 0$

$2(1) + 2 - (-1) = 5$

this is a solution!!

The set of solutions of a linear equation in $x$ and $y$ is a line in the $xy$-plane, so a solution of a linear system in $x$ and $y$ corresponds to a point of intersection between lines.

**Example 3.** Solve each linear system, and interpret the solution(s) geometrically.

(a) $$\begin{aligned} x+y&=1\\ 2x+y&=4 \end{aligned}$$

- add $-2\times$ eq 1 to eq 2:

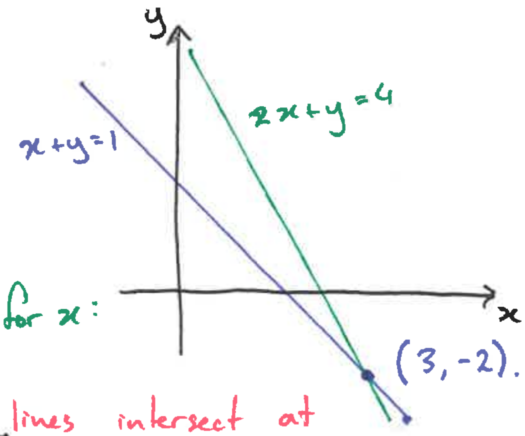
$$\begin{aligned} x+y&=1\\ -y&=2 \end{aligned}$$

- solve for $y$:

$$y=-2$$

- sub. into eq. 1 and solve for $x$:

$$\begin{aligned} x-2&=1\\ x&=3. \end{aligned}$$



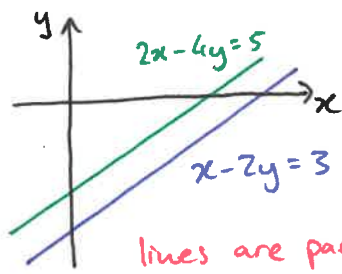

lines intersect at a <u>unique</u> point.

(b) $$\begin{aligned} x-2y&=3\\ 2x-4y&=5 \end{aligned}$$

- add $-2\times$ eq 1 to eq 2:

$$\begin{aligned} x-2y&=3\\ 0x+0y&=-1 \end{aligned}$$

no solutions (the system is <u>inconsistent</u>)




lines are parallel!!!

(c) $$\begin{aligned} 3x+y&=2\\ 9x+3y&=6 \end{aligned}$$

- add $-3\times$ eq 1 to eq 2:

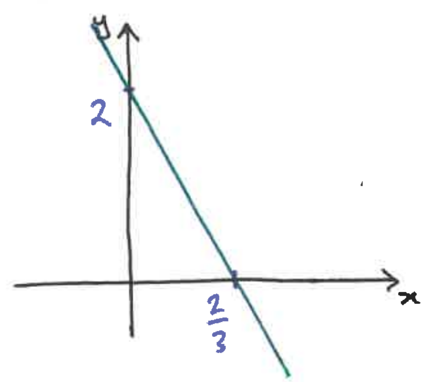
$$\begin{aligned} 3x+y&=2\\ 0&=0 \end{aligned}$$

- use a <u>parameter</u> for $y$:

Let $y=t$. Then $3x+t=2$,

so $x=-\frac{1}{3}t+\frac{2}{3}$.

The solution is $x=-\frac{1}{3}t+\frac{2}{3}$, $y=t$.




the lines are the same!! (coincident)

The set of solutions of a linear equation in three variables is a plane, so a solution of a linear system in three variables corresponds to a point of intersection between planes.

**Example 4.** Solve the linear system and interpret the solution(s) geometrically.

$$\begin{aligned} x + y - z &= 4 \\ 2x + 2y - 2z &= 8 \\ 4x + 4y - 4z &= 16 \end{aligned}$$

These three equations are equivalent, so the three planes are the same!!!

parametric sol$^{\underline{n}}$: let $y = s$, $z = t \Rightarrow x = -s + t + 4$

Solution is the plane $x = -s+t+4$, $y = s$, $z = t$.

More generally, a linear system is usually solved by performing elementary row operations on the augmented matrix for the system.

$$\underbrace{\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}}_{\text{linear system}} \qquad \underbrace{\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}}_{\text{augmented matrix}}$$

$$\begin{aligned} 2x + 0y - 4z &= -2 \\ 0x + 0y + z &= 2 \\ 0x + y + 0z &= 1 \end{aligned} \qquad \begin{bmatrix} 2 & 0 & -4 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

**Elementary row operations.**

1. Multiply a row by a nonzero constant.

eg. $$\begin{bmatrix} 2 & 0 & -4 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

2. Swap two rows.

eg. $$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

3. Add a multiple of one row to another row.

eg. $$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Gauss-Jordan elimination

$\hookrightarrow$ $x = 3$, $y = 1$, $z = 2$

**Example 5.** Solve the linear system and interpret the solution(s) geometrically.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\Big\downarrow R_2 \to \tfrac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 - 3R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{bmatrix}$$

$$\Big\downarrow R_3 \to -2R_3$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Big\downarrow R_1 \to R_1 - 2R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 + \tfrac{7}{2}R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Big\downarrow R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution is:

$x = 1, \quad y = 2, \quad z = 3.$

The three planes intersect at the point $(1,2,3)$.

## Section 1.2: Gaussian Elimination

**Objectives.**

- Identify matrices in row echelon form and reduced row echelon form.
- Use an augmented matrix in reduced row echelon form to write the solution for a linear system.
- Apply Gauss-Jordan elimination and Gaussian elimination to solve a linear system.
- Understand the relationship between numbers of unknowns, equations, and free variables.

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A matrix is in row echelon form when the following are true.

(a) If a row contains a nonzero number, then the first nonzero number in the row is a 1. (This is a leading 1.)

(b) Any rows that contain only zeroes are at the bottom of the matrix.

(c) If a row has a leading 1, then it is further to the right than the leading 1 in any higher row.

A matrix is in reduced row echelon form if it is in row echelon form and:

(d) If a column contains a leading 1, then every other number in the column is 0.

**Example 1.** Which of the matrices below are in row echelon form (ref)? Which are in reduced row echelon form (rref)? Which are neither?

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

r.e.f.

swap

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -7 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

r.r.e.f.

needs to be 1

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

neither

swap

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

r.r.e.f.

$$\begin{bmatrix} 1 & -4 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

r.e.f.

should be 1

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

neither

swap

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 2 & 4 & 0 & 8 \\ 1 & 0 & -5 & 2 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

r.r.e.f.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 7 \\ 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

r.r.e.f.

A variable corresponding to the leading 1 in some row is a leading variable. All other variables are free variables.

**Example 2.** Each augmented matrix below is in reduced row echelon form, and corresponds to a linear system in the variables $x$, $y$, and $z$. Find a solution for each linear system, identify the leading variables and the free variables, and describe the solution geometrically.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

leading 1s

leading vars: $x, y, z$ free vars: none

Solution is $x=1$, $y=2$, $z=3$.

This is the point $(1,2,3)$.

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 4 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free variable

leading vars: $x, y$ free vars: $z$

Let $z = t$. ← assign a parameter to each free var.

$x + 4t = -3 \Rightarrow x = -4t - 3$

$y + 2t = 0 \Rightarrow y = -2t$

The solution is $x = -4t-3$, $y = -2t$, $z = t$.

This is a line in three-dimensional space.

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

inconsistent system!!!

$0x + 0y + 0z = 1$.

leading vars: $x, y$ free vars: $z$

System has no solution.

leading 1

(d) $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free variables

leading vars: $x$ free vars: $y, z$

Let $y = s$, $z = t$.

Then $x - s - t = 0$, so $x = s + t$.

The sol$^{\underline{n}}$ is $x = s+t$, $y = s$, $z = t$.

This sol$^{\underline{n}}$ is a plane in three-dimensions.

Given an augmented matrix, an algorithm called <u>Gaussian elimination</u> can be used to find a matrix in row echelon form that has the same solutions.

**Gaussian elimination.**

1. Identify the leftmost column that contains a nonzero number.
2. If necessary, swap two rows so that the first number in this column is nonzero. Call this number $a$.
3. Multiply the top row by $\frac{1}{a}$ to create a leading 1.
4. Add multiples of the top row to each lower row so that every entry below the leading 1 is zero.
5. Cover the top row and repeat from Step 1.

**Example 3.** Apply Gaussian elimination to the augmented matrix below.

need a 1 at the top.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow R_1 \to \tfrac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\vdots$$

$$\downarrow R_2 \to -\tfrac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - 5R_2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

$$\downarrow R_3 \to 2R_3$$

$$\underbrace{\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{\text{matrix is in row echelon form.}}$$

leading vars. are $x_1, x_3, x_5$.

free vars. are $x_2, x_4$.

While Gaussian elimination will result in a matrix in row echelon form, Gauss-Jordan elimination is an extension that gives a matrix in reduced row echelon form.

**Gauss-Jordan elimination.**

1. Perform Gaussian elimination to obtain a matrix in row echelon form.
2. Starting from the bottom row and working upwards, identify the leading 1 in each row (if there is one).
3. Add multiples of this row to each higher row so that each entry above the leading 1 is a zero.

**Example 4.** Solve the linear system.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\qquad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

augmented matrix:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$\downarrow$ $R_2 \to R_2 - 2R_1$, $R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$\downarrow$ $R_2 \to -R_2$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 6 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$\downarrow$ $R_3 \to R_3 - 5R_2$, $R_4 \to R_4 - 4R_2$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$\downarrow$ $R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$ $R_3 \to \frac{1}{6} R_3$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$ $R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$ $R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_2 = r$, $x_4 = s$, $x_5 = t$. Then:

$x_1 = -3r - 4s - 2t$

$x_3 = -2s$

$x_6 = \frac{1}{3}$.

A linear system is homogeneous if each of the equations in the system is homogeneous.

i.e.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

A homogeneous linear system in the variables $x_1, x_2, \ldots, x_n$ always has the trivial solution

$$x_1 = x_2 = \cdots = x_n = 0.$$

(Any solution where at least one variable is nonzero is called a nontrivial solution.)

**Example 5.** Solve the linear system. *Hint: compare this system with the previous example.*

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 \phantom{{}-3x_6} &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \phantom{{}+4x_5} + 15x_6 &= 0 \\ 2x_1 + 6x_2 \phantom{{}-5x_3} + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right]$$

elementary row operations do not affect a column of zeros!!

$\downarrow$ row operations from Ex. 4.

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

r.r.e.f. from Ex. 4.

Let $x_2 = r$, $x_4 = s$, $x_5 = t$.

Then:

$x_1 = -3r - 4s - 2t$

$x_3 = -2s$

$x_6 = 0$

note: if $r = s = t = 0$, the solution is $(0,0,0,0,0,0)$.

trivial sol$^{\underline{n}}$.

**Theorem.** A homogeneous linear system with $n$ unknowns and $r$ nonzero rows in the reduced row echelon form of the augmented matrix has $n - r$ free variables.

**Theorem.** A homogeneous linear system with more unknowns that equations has infinitely many solutions.

An alternative to Gauss-Jordan elimination is to use Gaussian elimination followed by back-substitution.

**Gaussian elimination with back-substitution.**

1. Perform Gaussian elimination to obtain a matrix in row echelon form.
2. Write an equation for each leading variable in terms of the other variables.
3. Starting from the bottom, substitute each equation into the equations above it.
4. Replace each free variable with a parameter.

**Example 6.** Use back-substitution to solve the linear system in Example 4.

From Ex. 4.

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ x_3 + 2x_4 + 3x_6 = 1 \\ x_6 = 1/3 \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2x_3 - 2x_5 \\ x_3 = 1 - 2x_4 - 3x_6 \\ x_6 = 1/3 \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2x_3 - 2x_5 \\ x_3 = 1 - 2x_4 - 3\left(\frac{1}{3}\right) = -2x_4 \\ x_6 = \frac{1}{3} \end{cases}$$

$$\begin{cases} x_1 = -3x_2 + 2(-2x_4) - 2x_5 \\ \quad = -3x_2 - 4x_4 - 2x_5 \\ x_3 = -2x_4 \\ x_6 = 1/3 \end{cases}$$

The solution is:

$x_1 = -3r - 4s - 2t$
$x_2 = r$
$x_3 = -2s$
$x_4 = s$
$x_5 = t$
$x_6 = 1/3$ .

**Discussion.** For each augmented matrix below, identify the number of solutions for the corresponding linear system.

$$\begin{bmatrix} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variable, system is consistent $\Rightarrow$ infinite solutions

$$\begin{bmatrix} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

no free variable, consistent system $\Rightarrow$ one solution.

$$\begin{bmatrix} 1 & 2 & 6 & 0 & -15 \\ 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$0 = 1$!!!!

no solutions (inconsistent system)

## Section 1.3: Matrices and Matrix Operations

**Objectives.**

- Recognize a rectangular array of numbers as a matrix.
- Understand basic terminology and notation used for matrices.
- Apply the operations of matrix addition, subtraction, and multiplication correctly.
- Compute a linear combination of matrices.
- Find the transpose and the trace of a matrix.

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An $m \times n$ matrix is a rectangular array of numbers with $m$ rows and $n$ columns. A square matrix of order $n$ is a matrix with $n$ rows and $n$ columns.

"main diagonal"

eg. $\begin{bmatrix} 1 & 3 & 2 & 0 \\ 2 & 2 & 1 & 2 \\ 1 & -2 & 4 & 6 \end{bmatrix}$ is a 3x4 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a square matrix of order 2.

A matrix with one row is called a row vector (or row matrix). A matrix with one column is called a column vector (or column matrix).

row vector: $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ column vector: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Two matrices are equal if they have the same size and their corresponding entries are equal. If two matrices have the same size, then their sum (or difference) is found by adding (or subtracting) corresponding entries. A matrix can be multiplied by a scalar by multiplying each entry by the scalar.

**Example 1.** Simplify each expression.

(a) $$\begin{bmatrix} 3 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 \\ 4 & 2 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 3 & 2 & 1 \\ 1 & -5 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 & 7 \\ 1 & 2 & 3 & 0 \\ 5 & -3 & 9 & 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 7 & 3 & 0 & 2 \\ 5 & -1 & 2 & \cancel{-}1 \\ -2 & 2 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & -5 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & 1 \\ 5 & -2 & 0 & \cancel{-3} \\ -3 & 7 & -6 & -4 \end{bmatrix} -1$$

(c) $$2\begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -4 \\ 2 & -2 & 0 \\ -6 & 4 & 8 \end{bmatrix}$$

different sizes!!!

note: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is undefined!!!

If $A$ is an $m \times r$ matrix and $B$ is an $r \times n$ matrix, then the product $AB$ is an $m \times n$ matrix. The entry in the $i$th row and $j$th column of $AB$ is found by multiplying each entry in the $i$th row of $A$ by the corresponding entry in the $j$th column of $B$ and adding the results.

$$\begin{array}{c} \\ \\ i\text{th row} \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{array}{c} \qquad\qquad \downarrow j\text{th column} \\ \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & \vdots & & \\ b_{r1} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \end{array} \qquad (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

**Example 2.** Compute each product below (if possible).

(a) $$\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{-1} \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} \cancel{3} & 4 & -1 \\ \cancel{-2} & 2 & 5 \\ \cancel{1} & 3 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 6 \\ 9 & 17 & 7 \end{bmatrix}$$

$(1)(3) + (2)(-2) + (-1)(1) = -2$ $\qquad$ $(1)(4) + (2)(2) + (-1)(3) = 5$

$(2)(3) + (0)(-2) + (3)(1) = 9$ $\qquad$ $(2)(4) + (0)(2) + (3)(3) = 17$ $\qquad$ etc.

(b) $$\begin{bmatrix} 3 & 4 & -1 \\ -2 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix} = \text{undefined.}$$

$3 \times \underline{3}$ matrix $\qquad$ $\underline{2} \times 3$ matrix

dimensions do <u>not</u> match

(c) $$\begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 & 5 \\ 3 & 2 & 1 & -4 \\ -3 & 0 & 3 & 6 \end{bmatrix}$$

A matrix can be partitioned into <u>submatrices</u> by selecting certain rows and/or columns.

$$\left[\begin{array}{cc:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hdashline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{where} \quad A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}, \quad A_{21} = \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} a_{33} \end{bmatrix}$$

$$\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hdashline a_{21} & a_{22} & a_{23} \\ \hdashline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix} \quad \text{where} \quad \vec{r}_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}, \text{ etc.}$$

$$\left[\begin{array}{c:c:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{bmatrix} \quad \text{where} \quad \vec{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \text{ etc.}$$

Partitioning matrices into rows and columns allows some different strategies for matrix multiplication. This is particularly useful when only some rows and/or columns of the product are needed.

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$$

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} B = \begin{bmatrix} \vec{a}_1 B \\ \vec{a}_2 B \\ \vdots \\ \vec{a}_n B \end{bmatrix}$$

**Example 3.** Simplify each expression.

(a) $$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

compare with last column of Ex. 2(a).

(b) $$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -3 \end{bmatrix} = 1\begin{bmatrix} -1 & 0 & 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & -4 \end{bmatrix}.$$

compare with $2^{nd}$ row of Ex. 2(c).

If $A_1, A_2, \dots, A_n$ are matrices of the same size, and $c_1, c_2, \dots, c_n$ are scalars, then

$$c_1 A_1 + c_2 A_2 + \cdots + c_n A_n$$

is a linear combination of $A_1, A_2, \dots, A_n$.

When $B$ is a column vector, the product $AB$ is a linear combination of the columns of $A$.

**Example 4.** Simplify.

$$\begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y + z \\ -3x + 4y \\ -x + 8y + 5z \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} z$$

The last example suggests that we can express a linear system using matrix multiplication rather than an augmented matrix.

- linear system:

$$\begin{aligned} 2x + \phantom{4}y + \phantom{5}z &= 5 \\ -3x + 4y \phantom{{}+5z} &= 2 \\ -x + 8y + 5z &= 0 \end{aligned}$$

- augmented matrix:

$$\left[\begin{array}{cccc} 2 & 1 & 1 & 5 \\ -3 & 4 & 0 & 2 \\ -1 & 8 & 5 & 0 \end{array}\right]$$

equivalent ways of writing a linear system

- matrix equation:

$A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 0 \\ -1 & 8 & 5 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

If $A$ is an $m \times n$ matrix, then the transpose $A^T$ is the $n \times m$ matrix is obtained by swapping the rows and columns of $A$.

**Example 5.** Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 & 2 & 3 \\ -5 & 1 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & -5 \\ 2 & 1 \\ 3 & 6 \end{bmatrix}$$

(c) $C = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$$C^T = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

$$B^T = \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix}$$

(d) $D = \begin{bmatrix} 3 & 4 & 0 \\ 4 & 5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

$$D^T = \begin{bmatrix} 3 & 4 & 0 \\ 4 & 5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

note: $D^T = D$, so $D$ is symmetric

**Properties of transposes.**

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(kA)^T = k\,A^T$
4. $(AB)^T = B^T A^T$ note that the order is swapped!!!

The trace of a square matrix $A$, denoted by $\mathrm{tr}(A)$, is the sum of the entries on the main diagonal. (The trace is undefined for matrices that are not square.)

**Example 6.** Find the trace (if possible) of each matrix in the previous example.

tr $A$ and tr $B$ are undefined, because $A$ and $B$ are not square matrices.

$\mathrm{tr}(C) = 1+1+1 = 3$, $\mathrm{tr}(D) = 3+5+(-1) = 7$.

## Section 1.4: Inverses; Algebraic Properties of Matrices

**Objectives.**

- Learn the algebraic rules for matrix addition and multiplication.
- Understand zero matrices, identity matrices, and inverse matrices.
- Find the inverse of a $2 \times 2$ matrix.
- Use an inverse matrix to solve a linear system.
- Compute powers of matrices and matrix polynomials.

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Many of the rules for matrix algebra will be familiar from previous mathematics classes.

**Properties of matrix algebra.** Lower case letters refer to scalars; upper case letters refer to matrices.

1. $A + B =$ $B + A$
2. $A + (B + C) =$ $(A + B) + C$
3. $A(BC) =$ $(AB)C$
4. $A(B \pm C) =$ $AB \pm AC$
5. $(A \pm B)C =$ $AC \pm BC$
6. $a(B \pm C) =$ $aB \pm aC$
7. $(a \pm b)C =$ $aC \pm bC$
8. $a(bC) =$ $(ab)C$
9. $a(BC) =$ $(aB)C = B(aC)$

Notice however that **matrix multiplication is <u>not</u> commutative**. That is, $AB \neq BA$ in general.

**Example 1.** Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. Compute $AB$ and $BA$.

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

not equal!!!

$$BA = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{2\times 2}$$

The $m \times n$ matrix where every entry is 0 is a zero matrix and is denoted by $0_{m \times n}$.

eg. $0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , $0_{1\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

**Properties of zero matrices.**

1. $A \pm 0 = A$
2. $A - A = 0$
3. $\underset{\text{scalar}}{0}A = \underset{\text{matrix}}{0}$
4. If $cA = 0$, then either $c = 0$ or $A = 0$.

The last property listed above is called the zero-product principle. This is **not** true for matrix multiplication, as shown in the previous example.

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0_{2\times 2},$$ but neither factor is zero.

It is also incorrect to 'cancel' factors in a matrix product.

**Example 2.** Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 3 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 3 \\ -1 & -2 \end{bmatrix}$. Compute $AB$ and $AC$.

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Thus $AB = AC$, but $B \neq C$.

A square matrix with 1 on the main diagonal and 0 everywhere else is called an identity matrix. This is denoted by either $I$ or $I_n$ (to specify the size of the matrix).

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , etc.

**Properties of identity matrices.** Let $A$ be an $m \times n$ matrix.

1. $AI_n = A$
2. $I_m A = A$

**Example 3.** Confirm the properties above for the matrix $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$.

$$AI = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} = A.$$

If $A$ is a square matrix, and $B$ is a square matrix such that $AB = BA = I$, then we call $A$ an invertible matrix (or nonsingular matrix) and we call $B$ an inverse of $A$.

**Example 4.** Show that $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

$$AB = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$AB = I$ and $BA = I$, so $B = A^{-1}$.

If $A$ does not have an inverse, then $A$ is not invertible (or singular).

**Example 5.** Show that $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ is a singular matrix. guess: $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Suppose $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus:
$$\begin{aligned} 1a + 0c &= 1 &\Rightarrow\ & a = 1 \\ 1b + 0d &= 0 &\Rightarrow\ & b = 0 \\ 2a + 0c &= 0 &\Rightarrow\ & a = 0 \end{aligned}$$

contradiction!!!

We have found a contradiction, so $A$ has no inverse.

**Example 6.** Show that if $B$ and $C$ are both inverses of $A$, then $B = C$.

By assumption, $AB = I$ and $CA = I$. Then:

$$B = IB = (CA)B = C(AB) = CI = C.$$

The previous example shows that if $A$ is invertible then its inverse is unique. We denote this inverse by $A^{-1}$.

eg. $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -5 \\ -3 & 2 \end{bmatrix}$. (from Ex. 4).

**Example 7.** Show that if $A$ and $B$ are both invertible and have the same size, then $(AB)^{-1} = B^{-1}A^{-1}$.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Thus $AB$ is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(The quantity $ad - bc$ is called the <u>determinant</u> of $A$. We study determinants in Chapter 2.)

**Example 8.** Decide whether each matrix is invertible, and find the inverse if possible.

(a) $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$\det(A) = (6)(1) - (-3)(-2) = 6 - 6 = 0$

$A$ is <u>not</u> invertible. (ie. $A$ is singular)

(b) $B = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}$

$\det(B) = (5)(1) - (1)(3) = 5 - 3 = 2.$

$\det B \neq 0$, so $B$ has an inverse!!!

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}.$$

Recall that a linear system can be written in the form $A\vec{x} = \vec{b}$. If the coefficient matrix $A$ is invertible, then the linear system can be solved by <u>multiplying both sides of the matrix equation by $A^{-1}$</u>.

**Example 9.** Solve the linear system.

$$5x + y = 2$$
$$3x + y = -2$$

if $A\vec{x} = \vec{b}$, then

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

$$\begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -8 \end{bmatrix}.$$

ie. $x = 2$, $y = -8$.

A square matrix can be raised to any nonnegative integer power.

$$A^0 = I, \quad A^1 = A, \quad A^2 = AA, \quad A^3 = AAA, \ldots$$

An invertible matrix can be raised to any integer power (positive or negative).

$$A^{-n} = (A^{-1})^n.$$

**Powers of invertible matrices.** Let $A$ be invertible, $n$ be an integer, and $k$ be a nonzero scalar.

1. $A^{-1}$ is invertible, and $(A^{-1})^{-1} =$ $A$
2. $A^n$ is invertible, and $(A^n)^{-1} =$ $(A^{-1})^n$
3. $kA$ is invertible, and $(kA)^{-1} =$ $k^{-1}A^{-1} = \frac{1}{k}A^{-1}.$

If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial and $A$ is a square matrix, then

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

**Example 10.** Let $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ and let $p(x) = x^2 - x + 3$.

(a) Compute $A^3$.

$$A^3 = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}.$$

(b) Compute $p(A)$.

$$p(A) = A^2 - A + 3I = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 8 & 3 \end{bmatrix}.$$

Recall that the transpose of a matrix is found by swapping the rows and the columns of the matrix.

**Example 11.** Show that if $A$ is invertible, then $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$.

$$A^T(A^{-1})^T = (AA^{-1})^T = I^T = I.$$

$$(A^{-1})^T A^T = (A^{-1}A)^T = I^T = I.$$

Therefore, $(A^T)^{-1} = (A^{-1})^T$.

## Section 1.5: Elementary Matrices and a Method for Finding $A^{-1}$

**Objectives.**

- Write each elementary row operation using matrix multiplication.
- Find the inverse of a given row operation.
- Use row operations to find the inverse of a matrix or show that the matrix is not invertible.

---

Recall the three elementary row operations:

- multiply one row by a constant
- swap two rows
- add a multiple of one row to another row.

Two matrices $A$ and $B$ are row equivalent if $A$ can be transformed into $B$ using elementary row operations.

eg. $\begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are row equivalent.

(i.e) $R_1 \leftrightarrow R_2$ , then $R_1 \to \frac{1}{4} R_1$

An elementary matrix is a matrix that can be obtained from an identity matrix using a single elementary row operation. Multiplication by an elementary matrix is the same as performing an elementary row operation.

**Example 1.** What elementary row operation is equivalent to calculating $EB$ for each matrix $E$ below?

(a) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← double row 2 of B.

$R_2 \longrightarrow 2R_2$

(c) $E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Swap row 1 and row 3.

$R_1 \longleftrightarrow R_3$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← add -3 times row 1 to row 2

$R_2 \to R_2 - 3R_1$

eg. $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

(d) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← multiply row 1 by 1.

$R_1 \to R_1$

Each elementary row operation can be reversed by applying another elementary row operation.

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 5R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Example 2.** Find an elementary $3 \times 3$ matrix that corresponds to each row operation, and find an elementary row operation that reverses each row operation.

| | elementary matrix | <u>inverse</u> row operation |
|---|---|---|
| (a) multiply row 3 by $-\frac{1}{5}$ <br> i.e. $R_3 \to -\frac{1}{5}R_3$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{5} \end{bmatrix}$ | $R_3 \to -5R_3$ <br> i.e. multiply by reciprocal. |
| (b) swap row 1 and row 2 <br> i.e. $R_1 \leftrightarrow R_2$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $R_1 \leftrightarrow R_2$ <br> i.e. swap rows again!!! |
| (c) add 4 times row 2 to row 1 <br> i.e. $R_1 \to R_1 + 4R_2$ | $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $R_1 \to R_1 - 4R_2$ <br> i.e. ~~subtract~~ <u>sub</u>tract $4R_2$ from $R_1$. |

**Equivalence Theorem.** If $A$ is an $n \times n$ matrix, then the following statements are equivalent.

1. $A$ is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution. ← <u>note</u>: $A\vec{x} = \vec{0}$ <u>always</u> has the solution $\vec{x} = \vec{0}$. If $A$ is invertible, then $\vec{x} = \vec{0}$ is the <u>only</u> solution.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ can be written as a product of elementary matrices.

The Equivalence Theorem says that if $A$ is invertible then there is a sequence of elementary row operations that reduces $A$ to $I_n$. The same sequence of row operations applied to $I_n$ results in the matrix $A^{-1}$.

**Inverting a matrix.** To find the inverse of an $n \times n$ matrix $A$:

1. Form the matrix $[A|I_n]$.
2. Apply elementary row operations to reduce $A$ to $I_n$.
3. The resulting matrix has the form $[I_n|A^{-1}]$.

**Example 3.** Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

• start with $[A\,|\,I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$\downarrow$ $R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$\downarrow$ $R_3 \to -R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$\downarrow$ $R_2 \to R_2 + 3R_3$, $R_1 \to R_1 - 3R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$\downarrow$ $R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

... finish with $[I\,|\,A^{-1}]$.

Therefore:

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

The algorithm for finding an inverse matrix can also be used to decide whether a matrix has an inverse.

**Example 4.** Determine whether $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 3 & 5 & 11 \end{bmatrix}$ is invertible and find the inverse if possible.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 5 & 11 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 10 & -2 & 1 & 0 \\ 0 & 2 & 20 & -3 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 - 2R_2$$

$$\text{uh-oh!!!} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 10 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

$A$ is not invertible, because we cannot reduce $A$ to $I_3$ using elementary row operations.

**Example 5.** Decide whether each homogeneous linear system has nontrivial solutions.

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad + 8x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix is invertible (Ex. 3), so the system has <u>only</u> the trivial solution.

(b)
$$\begin{aligned} x_1 + x_2 - 3x_3 &= 0 \\ 2x_1 + 3x_2 + 4x_3 &= 0 \\ 3x_1 + 5x_2 + 11x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 3 & 5 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The coefficient matrix is <u>not</u> invertible (Ex. 4), so there are nontrivial solutions.

## Section 1.6: More on Linear Systems and Invertible Matrices

**Objectives.**

- Use an inverse matrix to solve a linear system.
- Understand properties of invertible matrices.
- Determine all vectors $\vec{b}$ for which the linear system $A\vec{x} = \vec{b}$ is consistent.

---

**Theorem.** A linear system has either no solutions, exactly one solution, or an infinite number of solutions.

**Proof.** Suppose $\vec{x}_1$ and $\vec{x}_2$ are distinct solutions of $A\vec{x} = \vec{b}$.

Let $\vec{x}_0 = \vec{x}_1 - \vec{x}_2$. Then $\vec{x}_0 \neq \vec{0}$ ~~there~~ because $\vec{x}_1 \neq \vec{x}_2$. Also:

$$A\vec{x}_0 = A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}.$$

If $k$ is any scalar, then:

$$A(\vec{x}_1 + k\vec{x}_0) = A\vec{x}_1 + kA\vec{x}_0 = \vec{b} + k\cdot\vec{0} = \vec{b} + \vec{0} = \vec{b}.$$

That is, $\vec{x}_1 + k\vec{x}_0$ is a sol$^{\underline{n}}$ of $A\vec{x} = \vec{b}$ for any $k$.

Therefore, this system has infinitely many solutions.

**Theorem.** If $A$ is an invertible $n \times n$ matrix, and $\vec{b}$ is an $n \times 1$ column vector, then the linear system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

From the previous theorem, if $A$ is invertible then the system $A\vec{x} = \vec{b}$ can be solved by multiplying by $A^{-1}$.

**Example 1.** Solve the linear system.

$$\begin{aligned} 6x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + \phantom{2}x_2 + \phantom{3}x_3 &= 0 \\ 10x_1 + 3x_2 + 4x_3 &= -1 \end{aligned} \implies \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

The inverse of $A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$.

Thus:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ 4 \end{bmatrix}.$$

Sometimes we may want to solve several linear systems that have the same coefficient matrix $A$. For instance, supose that we want to solve all of the systems:

$$A\vec{x} = \vec{b}_1, \quad A\vec{x} = \vec{b}_2, \quad \dots, \quad A\vec{x}_k = \vec{b}_k.$$

If $A$ is invertible, then the solutions can be found using matrix multiplication.

$$\vec{x}_1 = A^{-1}\vec{b}_1, \quad \vec{x}_2 = A^{-1}\vec{b}_2, \quad \dots, \quad \vec{x}_k = A^{-1}\vec{b}_k$$

An alternate approach (which also works when $A$ is singular!) is to solve the systems at the same time by row-reducing the augmented matrix

$$\left[ A \mid \vec{b}_1 \mid \vec{b}_2 \mid \cdots \mid \vec{b}_k \right] \xrightarrow{\text{row operations}} \left[ I \mid \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_k \right].$$

**Example 2.** Solve the linear systems.

(a)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ x_2 - 2x_3 &= -2 \\ 2x_1 - 3x_2 + 2x_3 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 1 \\ x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 + 2x_3 &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 2 & -3 & 2 & 4 & -1 \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 3 & -6 & -6 & -3 \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 5 & 1 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

$$\Big\downarrow \; R_1 \to R_1 + 3R_2$$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & -2 & -1 & 4 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

part (a): $x_3 = t$, so $x_1 = -1 + 2t$, $x_2 = -2 + 2t$.

part (b): system is inconsistent, so there are no solutions.

Our *definition* of an inverse matrix $B = A^{-1}$ requires that both $AB = I$ and $BA = I$ are true. However, it is enough to know that at least one of these equations is true.

**Theorem.** Let $A$ and $B$ be a square matrices. If $AB = I$ or $BA = I$, then $B = A^{-1}$.

**Example 3.** Show that $B = A^{-1}$ for the matrices $A$ and $B$ below. (These are the matrices from Example 1.)

$$A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

From the Thm above, we only need to show $AB = I$ (or $BA = I$).

$$AB = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Therefore, $B = A^{-1}$. (also, $A = B^{-1}$).

**Equivalence Theorem.** If $A$ is an $n \times n$ matrix, then the following statements are equivalent.

1. $A$ is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ can be written as a product of elementary matrices.

from Sect. 1.5 (page 2)

5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.

new conditions.

**Theorem.** If $A$ and $B$ are square matrices and $AB$ is invertible, then both $A$ and $B$ are invertible.

**Proof.** Suppose $\vec{x}_0$ is a sol$^{\text{n}}$ to $B\vec{x} = \vec{0}$. Then

$$(AB)\vec{x}_0 = A(B\vec{x}_0) = A\vec{0} = \vec{0}.$$

Because $AB$ is invertible, the system $(AB)\vec{x} = \vec{0}$ has only the trivial solution. Thus $\vec{x}_0 = \vec{0}$. That is, $B\vec{x} = \vec{0}$ has only the trivial solution, so $B$ is invertible.

Thus $A = A(BB^{-1}) = \underline{(AB)B^{-1}}$ is invertible.

product of invertible matrices is invertible.

> **Problem.** Given an $m \times n$ matrix $A$, find all $m \times 1$ vectors $\vec{b}$ for which the linear system $A\vec{x} = \vec{b}$ is consistent.

If $A$ is invertible, this problem is easy. ($A\vec{x} = \vec{b}$ is consistent for every $m \times 1$ vector $\vec{b}$.) Otherwise, row operations can be used to determine which vectors $\vec{b}$ give consistent systems.

**Example 4.** What conditions must $b_1, b_2, b_3$ satisfy for the system below to be consistent?

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= b_1 \\ x_2 - 2x_3 &= b_2 \\ 2x_1 - 3x_2 + 2x_3 &= b_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 2 & -3 & 2 & b_3 \end{array}\right]$$

$$\Big\downarrow\ R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 3 & -6 & b_3 - 2b_1 \end{array}\right]$$

$$\Big\downarrow\ R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 - 3b_2 \end{array}\right]$$

For this system to be consistent, we need

$$b_3 - 2b_1 - 3b_2 = 0, \quad \text{so} \quad b_3 = 2b_1 + 3b_2.$$

## Section 1.7: Diagonal, Triangular, and Symmetric Matrices

**Objectives.**

- Identify diagonal, upper triangular, lower triangular, and symmetric matrices.
- Understand properties of diagonal, triangular, and symmetric matrices.

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Some matrices are easier to compute with than others, either because they contain a lot of zeroes or because or their symmetry. These matrices will be important in some of the topics we study later in this course.

A square matrix $A$ is:

- diagonal if the only nonzero entries are on the main diagonal.

  i.e. $a_{ij} = 0$ if $i \neq j$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- upper triangular if every entry below the main diagonal is zero.

  i.e. $a_{ij} = 0$ if $i > j$

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

- lower triangular if every entry above the main diagonal is zero.

  i.e. $a_{ij} = 0$ if $i < j$

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

- symmetric if $A = A^T$.

  i.e. $a_{ij} = a_{ji}$

**Example 1.** Identify each matrix as diagonal and/or upper triangular and/or lower triangular and/or symmetric.

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$$

diagonal, upper Δ, lower Δ, symm.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

symmetric

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ -3 & 8 & 3 \end{bmatrix}$$

lower Δ

$$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

upper Δ

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

none of these!! (not square).

$$\begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

upper Δ

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

symmetric

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

diagonal, upper Δ, lower Δ, symm.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

diagonal, upper Δ, lower Δ, symmetric

An $n \times n$ diagonal matrix can be written in the form

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

This matrix is invertible if and only if every entry on the main diagonal is nonzero, in which case the inverse is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

, provided none of the $d_i$ values is zero.

**Example 2.** Compute each inverse (if it exists!).

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{-1} =$ DNE!!!

If $k$ is a positive integer, then $D^k$ can be computed by raising each (nonzero) entry in $D$ to the power $k$.

**Example 3.** Simplify each expression (if possible!).

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & (\frac{1}{3})^3 & 0 \\ 0 & 0 & 1^3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & \frac{1}{27} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-4} = \begin{bmatrix} \frac{1}{16} & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{-4} =$ DNE!!!

cannot find $D^{-1}$, so we also cannot find $D^{-4}$.

Multiplication by a diagonal matrix is also relatively simple.

**Example 4.** Compute each product.

(a) $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 10 \\ 3 & 4 \end{bmatrix}$

i.e. multiply $R_1$ by $-3$
multiply $R_2$ by $2$
multiply $R_3$ by $1$

(b) $\begin{bmatrix} 4 & -1 & 2 \\ 8 & 1 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 4 \\ 8 & 4 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

i.e. multiply $C_1$ by $1$
" $C_2$ by $4$
" $C_3$ by $2$

**Properties of (upper) triangular matrices.** *Note: similar properties hold for lower triangular matrices.*

1. The transpose of an upper triangular matrix is lower triangular.
2. The product of two upper triangular matrices is upper triangular.
3. An upper triangular matrix is invertible if and only if every entry on the main diagonal is nonzero.
4. The inverse of an invertible upper triangular matrix is upper triangular.

**Example 5.** Suppose that $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Show that $A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } A^{-1} = \begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}.$$

(b) Compute $AB$ and $BA$. What do you notice?

$$AB = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

**Proof of 2.** Suppose $A, B$ are upper triangular, and let $C = AB$. If $i > j$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$
$$= \underbrace{a_{i1}b_{1j} + \cdots + a_{i(i-1)}b_{(i-1)j}}_{a_{i1}=0,\ \ldots,\ a_{i(i-1)}=0} + \underbrace{a_{ii}b_{ij} + \cdots + a_{in}b_{nj}}_{b_{ij}=0,\ \ldots,\ b_{nj}=0}$$
$$= \underline{0}.$$

Because $c_{ij} = 0$ when $i > j$, the matrix $C$ is upper triangular.

$A$ symmetric $\iff A = A^T$

**Properties of symmetric matrices.** If $A$ and $B$ are symmetric $n \times n$ matrices, and $k$ is a scalar, then:

1. $A^T$ is symmetric.
2. $A + B$ and $A - B$ are both symmetric.
3. $kA$ is symmetric
4. $AB$ is symmetric if and only if $AB = BA$.
5. If $A$ is invertible then $A^{-1}$ is symmetric.
6. If $A$ is invertible, then $AA^T$ and $A^TA$ are invertible.

**Proof of 2.**

$$(A+B)_{ij} = A_{ij} + B_{ij} = A_{ji} + B_{ji} = (A+B)_{ji}.$$

row $i$, col. $j$ of $A+B$

The next example illustrates property 4 above.

**Example 6.** Compute each product.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

AB is symmetric, so $AB = BA$.

One final observation is that for <u>any</u> matrix $A$, the products $AA^T$ and $A^TA$ are both symmetric.

$$(AA^T)^T = (A^T)^T A^T = AA^T, \text{ so } AA^T \text{ is symmetric.}$$

Swap order when using transpose!!!

**Example 7.** Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix}$. Confirm that both $AA^T$ and $A^TA$ are symmetric.

$$AA^T = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 19 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 3 & 7 \\ 3 & 1 & 3 \\ 7 & 3 & 10 \end{bmatrix}.$$

## Section 1.8: Introduction to Linear Transformations

**Objectives.**

- Understand an $m \times n$ matrix as a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Identity the standard basis vectors for $\mathbb{R}^n$ and the standard matrix of a transformation.
- Study some simple linear transformations.

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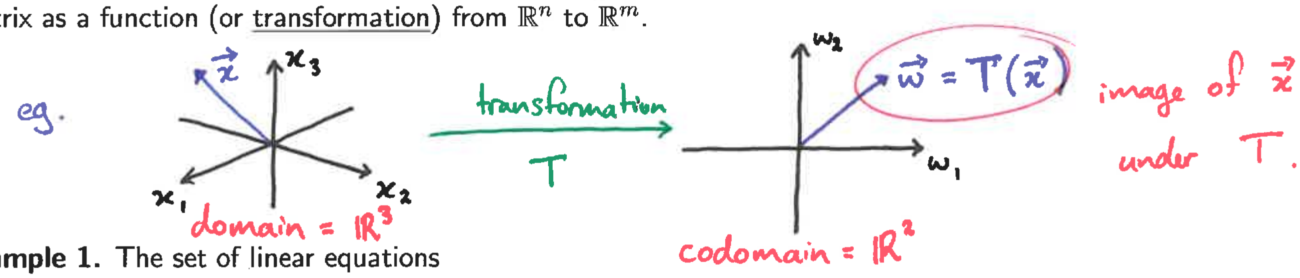
The set of all $n \times 1$ column vectors is denoted by $\mathbb{R}^n$. In this section, we interpret multiplication by an $m \times n$ matrix as a function (or transformation) from $\mathbb{R}^n$ to $\mathbb{R}^m$.




**Example 1.** The set of linear equations

$$\begin{aligned} w_1 &= \quad x_1 - 2x_2 + 4x_3 - 2x_4 \\ w_2 &= \ 3x_1 + \ x_2 - 2x_3 + \ x_4 \\ w_3 &= -6x_1 \qquad\ \ + \ x_3 - \ x_4 \end{aligned}$$

defines a linear transformation $T_A$ from $\mathbb{R}^4$ to $\mathbb{R}^3$.

(a) Express the transformation $T_A$ using matrix multiplication.

$$T_A(\vec{x}) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

or: $\vec{w} = T_A(\vec{x}) = A\vec{x}$

where $A = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix}$.

(b) Find the image of the vector $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ under the transformation $T_A$.

$$T_A\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ -4 \end{bmatrix}$$

Note: The linear transformation in this example can also be written in comma-delimited form as

$$T(x_1, x_2, x_3, x_4) = (\underbrace{x_1 - 2x_2 + 4x_3 - 2x_4}_{w_1}, \underbrace{3x_1 + x_2 - 2x_3 + x_4}_{w_2}, \underbrace{-6x_1 + x_3 - x_4}_{w_3}).$$

Two simple matrix transformations are the zero transformation/operator and the identity transformation/operator.

$T_0(\vec{x}) = 0\vec{x} = \vec{0}$

"zero transformation"

$T_I(\vec{x}) = I\vec{x} = \vec{x}$

"identity transformation"

**Properties of matrix transformations.** If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and $k$ is a scalar, then:

1. $T_A(\vec{0}) = \vec{0}$ $\longrightarrow$ the zero vector/origin is unchanged by a matrix transformation
2. $T_A(k\vec{u}) = k\,T_A(\vec{u})$ $\rightarrow$ "homogeneity"
3. $T_A(\vec{u}+\vec{v}) = T_A(\vec{u}) + T_A(\vec{v})$ $\rightarrow$ "additive property"

Not all transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are matrix transformations. For instance:

$$\begin{aligned} w_1 &= x_1 + x_2^2 \\ w_2 &= x_1 x_2 \end{aligned}$$

is not a matrix transformation.

"non linear terms"

However, a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies both homogeneity and the additivity property is a matrix transformation.

(More specifically, if these two properties are satisfied then $T$ is called a linear transformation. That is, every matrix transformation is a linear transformation, and every linear transformation is a matrix transformation.)

**Example 2.** Show that $T(x, y) = (x + 3y, 2x, 2x - y)$ is a linear transformation.

Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$. Then:

$$T(k\vec{u}) = T(ku_1, ku_2) = (ku_1 + 3ku_2, 2ku_1, 2ku_1 - ku_2)$$
$$= k(u_1 + 3u_2, 2u_1, 2u_1 - u_2) = k\,T(u_1, u_2) = k\,T(\vec{u}).$$

$$T(\vec{u} + \vec{v}) = T(u_1 + v_1, u_2 + v_2) = (u_1 + v_1 + 3(u_2 + v_2), 2(u_1 + v_1), 2(u_1 + v_1) - (u_2 + v_2))$$
$$= (u_1 + 3u_2, 2u_1, 2u_1 - u_2) + (v_1 + 3v_2, 2v_1, 2v_1 - v_2)$$
$$= T(\vec{u}) + T(\vec{v}).$$

$T$ satisfies homogeneity and the additive property, so $T$ is a linear transformation

**Theorem.** If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and $T_A(\vec{x}) = T_B(\vec{x})$ for every vector $\vec{x}$ in $\mathbb{R}^n$, then $A = B$.

As a consequence of this theorem, each linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ corresponds to exactly one $m \times n$ matrix, which we call the standard matrix for the transformation.

eg. the standard matrix in Ex. 1 is $A = \begin{bmatrix} 1 & -2 & 4 & -2 \\ 3 & 1 & -2 & 1 \\ -6 & 0 & 1 & -1 \end{bmatrix}$.

The standard basis vectors for $\mathbb{R}^n$ are the $n \times 1$ vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Every vector in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

eg. in $\mathbb{R}^3$: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$.

**Example 3.** Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$

or: $T(x_1, x_2) = (x_1 - x_2,\ 2x_1 + x_2,\ x_1 + 3x_2)$.

(a) Compute $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2-3 \\ 2(2)+3 \\ 2+3(3) \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 11 \end{bmatrix}.$$

← "image of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under $T$"

(b) Find the image of each standard basis vector in $\mathbb{R}^2$.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-0 \\ 2(1)+0 \\ 1+3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-1 \\ 2(0)+1 \\ 0+3(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

(c) Find the standard matrix for this linear transformation.

$$A = \left[\, T(\vec{e}_1) \mid T(\vec{e}_2) \,\right] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

**Example 4.** Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}-1\\-2\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = \begin{bmatrix}10\\0\end{bmatrix}.$$

(a) Find the standard matrix for $T$.

- write $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$.

$$\begin{bmatrix}1\\0\end{bmatrix} = a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}2\\2\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix} = c\begin{bmatrix}1\\-1\end{bmatrix} + d\begin{bmatrix}2\\2\end{bmatrix}$$

- solve for constants.

$$a = \tfrac{1}{2},\ b = \tfrac{1}{4} \qquad c = -\tfrac{1}{2},\ d = \tfrac{1}{4}.$$

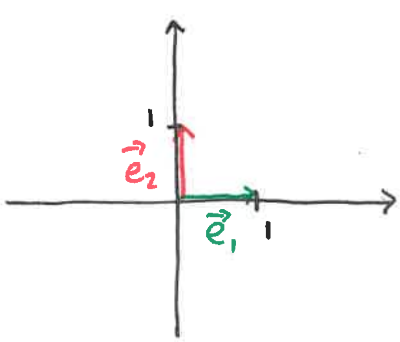
- find $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

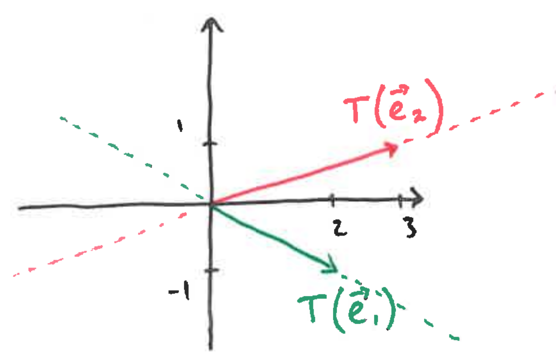
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\tfrac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}2\\2\end{bmatrix}\right) = \tfrac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \tfrac{1}{4}T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix}-1\\-2\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}10\\0\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = -\tfrac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) + \tfrac{1}{4}T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = -\tfrac{1}{2}\begin{bmatrix}-1\\-2\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}10\\0\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}.$$

$$A = \left[T(\vec{e}_1) \,\middle|\, T(\vec{e}_2)\right] = \begin{bmatrix}2 & 3\\-1 & 1\end{bmatrix}.$$

(b) Sketch a diagram showing each standard basis vector in $\mathbb{R}^2$, and another showing the <u>image</u> of each standard basis vector under the transformation $T$.

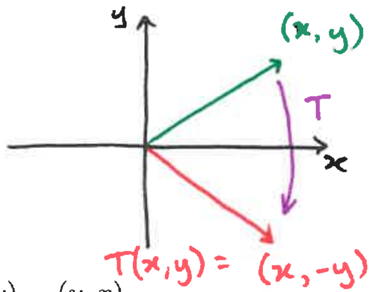


A linear transformation can be interpreted geometrically as a distortion of space that preserves straight lines. (The origin should also remain unchanged!) Some simple examples of these transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ include reflections, (orthogonal) projections, and rotations.

**Example 5.** For each transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, sketch a diagram showing a typical vector $\vec{x}$ and its image $T(\vec{x})$. Then describe the transformation and find the standard matrix for the transformation.

(a) $T(x, y) = (x, -y)$

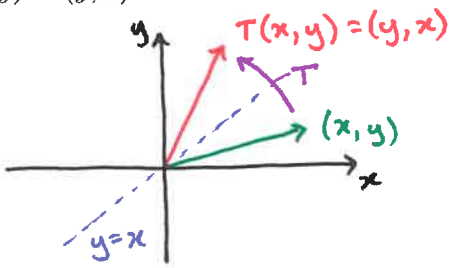



- reflection in $x$-axis.

$T(1,0) = (1,0)$, $T(0,1) = (0,-1)$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) $T(x, y) = (y, x)$

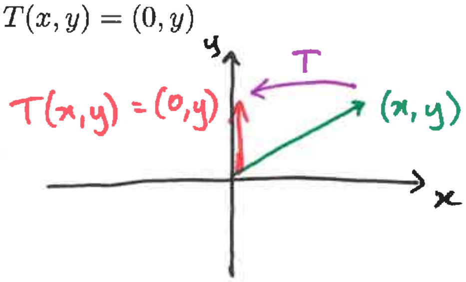



- reflection in the line $y = x$.

$T(1,0) = (0,1)$, $T(0,1) = (1,0)$.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $T(x, y) = (0, y)$




- (orthogonal) projection onto $y$-axis

$T(1,0) = (0,0)$, $T(0,1) = (0,1)$.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

**Example 6.** Describe each transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ and find the standard matrix for the transformation.

(a) $T(x, y, z) = (x, 0, z)$

- orthogonal projection onto $xz$-plane

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

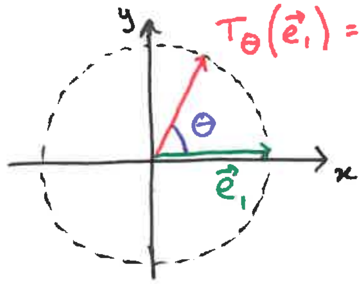
(b) $T(x, y, z) = (x, y, -z)$

- reflection in $xy$-plane

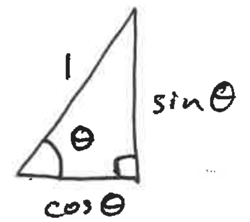
$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

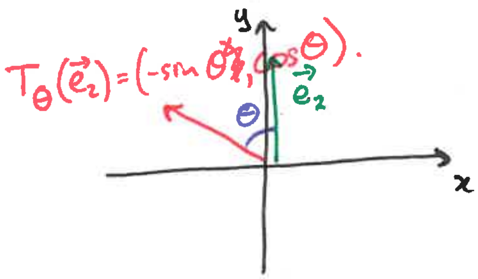
In $\mathbb{R}^2$, rotation about the origin by an angle $\theta$ is a linear transformation. We can find the standard matrix for this rotation by considering the image of the standard basis vectors.

$T_\theta(\vec{e}_1) = (\cos\theta, \sin\theta)$




$T_\theta(1,0) = (\cos\theta, \sin\theta)$




$T_\theta(0,1) = (-\sin\theta, \cos\theta)$.

$$\underbrace{R_\theta}_{\text{counterclockwise rotation by } \theta.} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

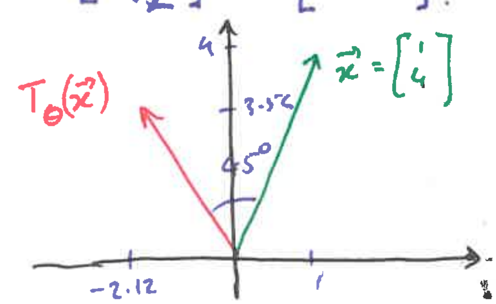
**Example 7.** Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represents a rotation of $45°$ about the origin.

(a) Find the standard matrix for the transformation.

$$R_\theta = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

(b) Find the image of $\vec{x} = (1, 4)$ under this transformation.

$$T_\theta(1,4) = R_\theta \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} -2.12 \\ 3.54 \end{bmatrix}.$$



## Section 2.1: Determinants by Cofactor Expansion

**Objectives.**

- Understand how to find minors and cofactors.
- Use minors and cofactors to compute the determinant of a square matrix.
- Find the determinant of a $3 \times 3$ matrix efficiently.

---

Recall that the determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

notation: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

We will use this to *inductively/recursively* define determinants for larger square matrices.

If $A = [a_{ij}]$ is a square matrix, then

- the minor of $a_{ij}$ is the determinant of the submatrix obtained from $A$ by deleting row $i$ and column $j$.

  $M_{ij}$

- the cofactor of $a_{ij}$ is $C_{ij} = (-1)^{i+j} M_{ij}$.

  $C_{ij}$

**Example 1.** Let $A = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \\ -1 & 8 & 2 \end{bmatrix}$.

(a) Find the minor of $a_{11}$ and the cofactor of $a_{11}$.

$$M_{11} = \det\left(\begin{bmatrix} 3 & 5 \\ 8 & 2 \end{bmatrix}\right) = 6 - 40 = \underline{-34.}$$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2(-34) = \underline{-34.}$$

(b) Find the minor of $a_{23}$ and the cofactor of $a_{23}$.

$$M_{23} = \begin{vmatrix} 2 & -1 \\ -1 & 8 \end{vmatrix} = 16 - 1 = \underline{15.}$$

$$C_{23} = (-1)^{2+3} M_{23} = (-1)^5(15) = \underline{-15.}$$

## Cofactor Expansion.

If $A$ is an $n \times n$ matrix, then the determinant of $A$ is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad \text{expansion along } i^{th} \text{ row.}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad \text{expansion along } j^{th} \text{ column.}$$

**Example 2.** Write out the cofactor expansion of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ along the first column.

$$\det(A) = \underset{\substack{\uparrow \\ a}}{a_{11}}\,\underset{\substack{\uparrow \\ (-1)^2 d}}{C_{11}} + \underset{\substack{\uparrow \\ c}}{a_{21}}\,\underset{\substack{\uparrow \\ (-1)^3 b}}{C_{21}} = ad + c(-b) = \underline{ad - bc.}$$

**Example 3.** Find the determinant of the matrix $B = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{bmatrix} \end{array}$.

$$\det(B) = 1\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & -2 \\ 4 & 5 \end{vmatrix}$$

$$= \big((-2)(2) - (3)(5)\big) - 3\big((2)(2) - (3)(4)\big) + 0\big((2)(5) - (-2)(4)\big)$$

$$= -19 - 3(-8) + 0 \quad = \underline{5.}$$

lots of zeros!!!

**Example 4.** Find the determinant of the matrix $C = \begin{array}{c} \begin{array}{cccc} + & - & + & \phantom{4} \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 1 & 0 & 5 & 2 \\ -1 & 1 & 0 & 3 \end{bmatrix} \end{array}$.

$$\det(A) = 0\begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 & 4 \\ 0 & 1 & -3 \\ -1 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 & 4 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 5\left(2\begin{vmatrix} 1 & -3 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} -1 & 4 \\ 1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix}\right)$$

$$= 5\big(2(3+3) - 0 - (3-4)\big)$$

$$= \underline{65.}$$

**Theorem.** The determinant of an upper triangular matrix, a lower triangular matrix, or a diagonal matrix is the product of the diagonal entries.

**Example 5.** Show that the theorem above holds for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$.

- use cofactor expansion along column 1.

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix} - 0 + 0 - 0$$

$$= a_{11}\left(a_{22} \begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix} - 0 + 0\right)$$

$$= \underline{a_{11} a_{22} a_{33} a_{44}.}$$

Finding determinants can be very time-consuming, especially for large matrices. There is an efficient method for computing the determinant of a $3 \times 3$ matrix (without using cofactor expansion) that is similar to how we compute the determinant of a $2 \times 2$ matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$-bc$ $\quad +ad$

**Example 6.** Find the determinant of $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{bmatrix}$.

$$\left|\begin{matrix} 1 & 3 & 0 \\ 2 & -2 & 3 \\ 4 & 5 & 2 \end{matrix}\right| \begin{matrix} 1 & 3 \\ 2 & -2 \\ 4 & 5 \end{matrix}$$

$-0 \quad -15 \quad -12 \quad +(-4) \quad +36 \quad +0$

$$\det(B) = [-4 + 36 + 0] + [-0 - 15 - 12]$$

$$= \underline{5.}$$

**Example 7.** Find all values of $\lambda$ for which the determinant of $A = \begin{bmatrix} \lambda+1 & 1 \\ 4 & \lambda-2 \end{bmatrix}$ is 0.

$$\det(A) = (\lambda+1)(\lambda-2) - 4 = \lambda^2 - \lambda - 2 - 4 = \lambda^2 - \lambda - 6 = (\lambda+2)(\lambda-3).$$

Thus $\det(A) = 0$ if $\lambda = -2$ or $\lambda = 3$.

**So ... what is a determinant?**

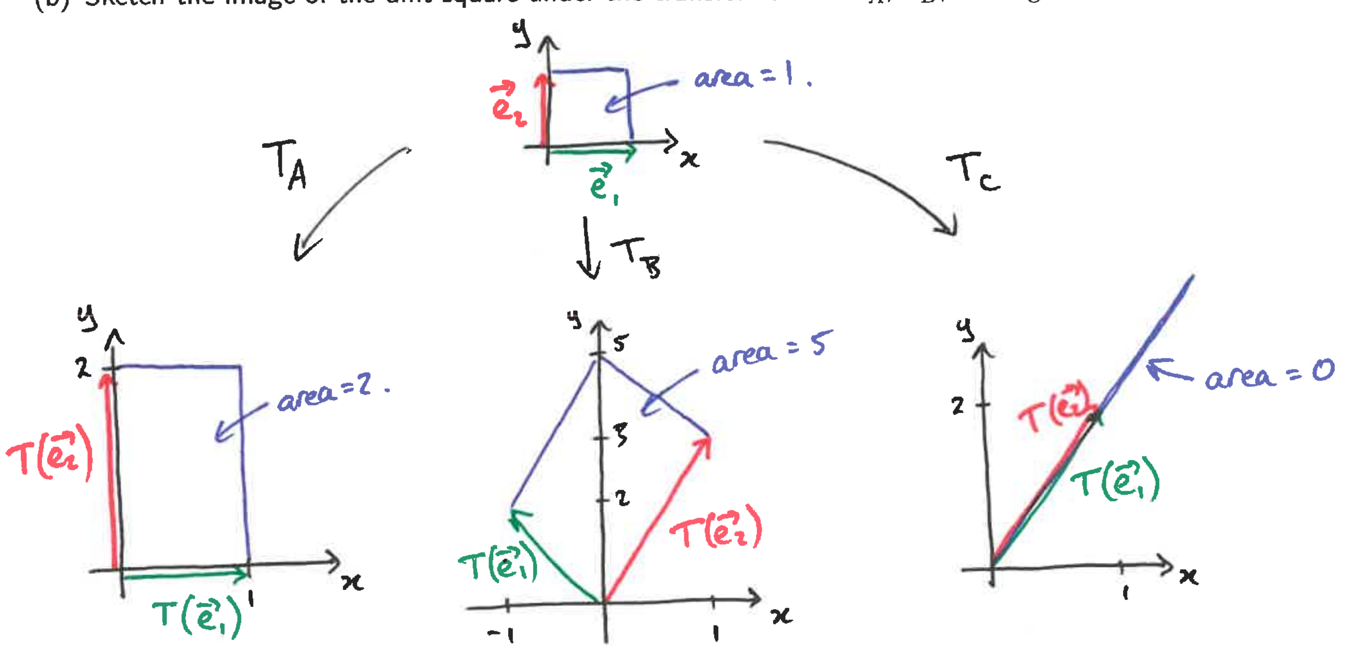
In some sense, the determinant of a square matrix $A$ is a scaling factor for the linear transformation $T_A$. For instance, if $A$ is a $2 \times 2$ matrix, then (the absolute value of) $\det A$ is the area of the parallelogram obtained by applying $T_A$ to the unit square.

**Example 8.** Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Find $\det A$, $\det B$, and $\det C$.

$\det(A) = 2$, $\det(B) = -5$, $\det(C) = 0$.

(b) Sketch the image of the unit square under the transformations $T_A$, $T_B$, and $T_C$.




(c) Compare the determinants in part (a) with each image in part (b).

**Section 2.2: Evaluating Determinants by Row Reduction**

**Objectives.**

- Understand how elementary row operations affect determinants.
- Use row reduction to compute determinants.
- Introduce column operations and apply them to compute determinants.

---

The "cofactor expansion" method for finding determinants leads to some useful observations.

**Theorem.** Let $A$ be a square matrix. If $A$ has a row (or column) of zeros, then $\det A = 0$.

eg. $\det\left(\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}\right) = 0$ , $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0.$ ← notation means "determinant"

**Theorem.** Let $A$ be a square matrix. Then $\det A = \det A^T$.

why? cofactor expansion on the $i^{th}$ row of $A$ is the same as cofactor on the $i^{th}$ column of $A^T$.

**Theorem.** Let $A$ be a square matrix.

(a) If $B$ is obtained by multiplying a row (or column) of $A$ by a scalar $k$, then $\det B = k \det A$.

eg. $$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(b) If $B$ is obtained by swapping two rows (or columns) of $A$, then $\det B = -\det A$.

eg. $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} b & a \\ d & c \end{vmatrix}.$$

(c) If $B$ is obtained by adding a multiple of one row of $A$ to another (or a multiple of one column of $A$ to another), then $\det B = \det A$.

eg. $$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

**Theorem.** Let $E$ be an $n \times n$ elementary matrix.

(a) If $E$ is obtained by multiplying a row of $I_n$ by a scalar $k$, then $\det E = k$.

(b) If $E$ is obtained by swapping two rows of $I_n$, then $\det E = -1$.

(c) If $E$ is obtained by adding a multiple of one row of $I_n$ to another, then $\det E = 1$.

eg. $\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = k$ , $\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = -1$, $\det\left(\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1$.

**Theorem.** Let $A$ be a square matrix. If two rows (or two columns) of $A$ are proportional, then $\det A = 0$.

eg. $\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = 0$ , $\det\left(\begin{bmatrix} 1 & 3 & -1 \\ 3 & 9 & -3 \\ 3 & 0 & 4 \end{bmatrix}\right) = 0$

$C_2 = 2C_1$ $R_2 = 3R_1$

**Example 1.** Find each determinant.

(a) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3} = - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -2.$ ← diagonal matrix!!!

(b) $\begin{vmatrix} 1 & 0 & -4 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix} = -3.$ ← upper triangular!!! [alternative: $R_1 \to R_1 + 4R_3$].

$R_2 \to R_2 - R_1$

(c) $\begin{vmatrix} 1 & 7 & 3 & 0 & 2 \\ 0 & -1 & -5 & 0 & 0 \\ -1 & 2 & -2 & 0 & -2 \\ 3 & 0 & 5 & 1 & 6 \\ 1 & 0 & 0 & 0 & 2 \end{vmatrix} = 0$, because $C_5 = 2C_1$

**Example 2.** Use row reduction to compute each determinant.

(a)
$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

• swap $R_1$ and $R_2$ → multiply by $-1$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$R_1 \to \frac{1}{3}R_1$ → take factor of 3 outside the determinant.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$R_3 \to R_3 - 2R_1$ → determinant does not change!!!

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$R_3 \to R_3 - 10R_2$ → determinant does not change!!!

$$= -3(1)(1)(-55)$$

$$= \underline{165.}$$

(b)
$$\begin{vmatrix} -1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 7 \\ -1 & 2 & 4 & 14 \\ 0 & 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & -2 & 2 & 8 \\ 0 & 2 & 4 & 6 \end{vmatrix}$$

$R_3 \to R_3 - R_1$

cofactor expansion along column 1.

$$= (-1)\begin{vmatrix} 0 & 1 & 7 \\ -2 & 2 & 8 \\ 2 & 4 & 6 \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= -\begin{vmatrix} 0 & 1 & 7 \\ 0 & 6 & 14 \\ 2 & 4 & 6 \end{vmatrix}$$

cofactor expansion along column 1.

$$= -2\begin{vmatrix} 1 & 7 \\ 6 & 14 \end{vmatrix}$$

$$= -2(14 - 42)$$

$$= \underline{56.}$$

We can also use column operations to simplify determinant calculations.

**Example 3.** Find the determinant of each matrix.

(a) $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 7 & 0 & -4 \\ 1 & -3 & 3 & 2 \\ 2 & 6 & -5 & 3 \end{bmatrix}$

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -2 & 7 & 0 & 4 \\ 1 & -3 & 3 & 2 \\ 2 & 6 & -5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 5 & 0 & 0 \\ 1 & -2 & 3 & 0 \\ 2 & 8 & -5 & -1 \end{vmatrix}$$

$C_2 \to C_2 + C_1$

$C_4 \to C_4 - 2C_1$

$$= (1)(5)(3)(-1)$$

$$= -15.$$

(b) $B = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$

$$\det(B) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix}$$

$R_1 \to R_1 - 3R_2$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 3R_2$

$$= \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

cofactor expansion!!!

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$R_3 \to R_3 + R_1$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

cofactor expansion!!!

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

$$= -18.$$

**Determinants and Solutions of Linear Systems.**

In Sections 1.5 and 1.6, we learned about the "Equivalance Theorem", which gives several conditions that are equivalent to a linear system having a unique solution. We can now add a condition involving determinants.

**Equivalence Theorem.** If $A$ is an $n \times n$ matrix, then the following statements are equivalent.

1. $A$ is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ can be written as a product of elementary matrices.

Section 1.5

5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.

Section 1.6

7. $\det A \neq 0$

Section 2.3.

**Example 4.** Which of the following matrices is invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & -5 \\ 2 & 0 & 2 \end{bmatrix}$$

$\det A = 0$, so $A$ is not invertible

$\det B \neq 0$, so $B$ is invertible

$\det C \neq 0$, so $C$ is invertible (eg. swap $R_1$ and $R_2$)

$\det D = 0$ ($R_3 = 2R_1$) so $D$ is not invertible!!!

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 & 1 & 5 \\ -4 & 0 & 4 & 1 \\ 0 & 0 & 6 & 2 \\ 2 & 0 & -3 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

can make triangular by swapping rows
$\det F \neq 0$
$\Rightarrow F$ is invertible.

$\det G = 0$
(column of zeros)
$\Rightarrow G$ is not invertible.

$R_3 = R_1 + R_2$,
so $\det H = 0$
$\Rightarrow H$ is not invertible

## Section 2.3: Properties of Determinants; Cramer's Rule

**Objectives.**

- Understand how determinants interact with matrix operations.
- Introduce the adjoint of a square matrix.
- Apply Cramer's Rule to solve a linear system.

---

We have several methods for finding the determinant of a matrix. We now want to find ways to deal with determinants of expressions such as $kA$, $A+B$, $AB$, and $A^{-1}$.

> If $A$ is an $n \times n$ matrix, and $k$ is a scalar, then $\det(kA) = k^n \det A$.

**Example 1.** Confirm the property above for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the scalar $k$.

$$\det(kA) = \det\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = (ka)(kd) - (kb)(kc)$$

$$= k^2(ad-bc) = k^2 \det A.$$

> If $A$ and $B$ are square matrices of the same size, then $\det(AB) = (\det A)(\det B)$.

**Example 2.** Confirm the property above for the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$.

$\det A = 4-(-4) = 8$, $\det B = 6-5 = 1$, $(\det A)(\det B) = 8$.

$AB = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -7 & 12 \\ -10 & 16 \end{bmatrix}$, $\det(AB) = -112-(-120) = 8$.

equal!!!

> If $A$ is an invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

**Example 3.** Suppose that $A$ is invertible. Use $\det(AB) = (\det A)(\det B)$ to prove that $\det(A^{-1}) = \dfrac{1}{\det A}$.

If $A$ is invertible, then $A^{-1}$ exists and

$\det I = \det(AA^{-1}) = (\det A)(\det A^{-1})$, so $1 = (\det A)(\det A^{-1})$.

Therefore, $\det(A^{-1}) = \dfrac{1}{\det A}$.

For most pairs of matrices, the determinant of the sum is **not** the sum of the determinants.

| In general, $\det(A+B) \neq \det A + \det B$. |
|---|

**Example 4.** Confirm the property above for the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$.

$\det A = 8$, $\det B = 1$, $\det A + \det B = 9$.

$A+B = \begin{bmatrix} -1 & 4 \\ 5 & 0 \end{bmatrix}$, $\det(A+B) = -20$.

not equal!!!

The one situation where the sum of two determinants is useful is when two matrices are almost identical.

**Theorem.** Let $A$, $B$, and $C$ be square matrices that differ only in row $i$, and suppose that the $i$th row of $C$ is the sum of the $i$th row of $A$ and the $i$th row of $B$. Then $\det C = \det A + \det B$.

Why? cofactor expansion!!!

cofactor exp. along row $i$.

$$\det C = \overbrace{c_{i1}C_{i1} + c_{i2}C_{i2} + \cdots + c_{in}C_{in}} = (a_{i1}+b_{i1})C_{i1} + (a_{i2}+b_{i2})C_{i2} + \cdots + (\cdots)$$

$$= \underbrace{a_{i1}C_{i1} + \cdots + a_{in}C_{in}} + \underbrace{b_{i1}C_{i1} + \cdots + b_{in}C_{in}} = \det A + \det B.$$

A, B, C have the same cofactors along row $i$.

**Example 5.** Confirm this theorem for the matrices $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

$$\det A = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} = 2(-3-4) = -14.$$

$$\det B = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 3(2-4) = -6.$$

$$\det C = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 2 & 0 \end{vmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{vmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & -2 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = 2(-6-4) = -20.$$

Thus $\det C = \det A + \det B$.

also called "adjugate"

The (classical) adjoint of a square matrix $A$ is formed by transposing the matrix of cofactors.

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

**Example 6.** Find the adjoint of $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix}$.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} = -2 \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} = -(-6) = 6 \qquad C_{13} = \cdots = -6$$

$$C_{21} = \cdots = 1 \qquad C_{22} = \cdots = -3 \qquad C_{23} = 0$$

$$C_{31} = 0 \qquad C_{32} = -6 \qquad C_{33} = 6.$$

$$\operatorname{adj} A = \begin{bmatrix} -2 & 6 & -6 \\ 1 & -3 & 0 \\ 0 & -6 & 6 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix}.$$

A useful application of the adjoint matrix is finding an inverse.

**Theorem.** If $A$ is an invertible matrix, then $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$.

from Ex. 6:

$$A(\operatorname{adj} A) = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

**Example 7.** Find the inverse of the matrix $A$ in the previous example.

$$\det A = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{vmatrix} = -6$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{-6}\begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -6 \\ -6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & 0 \\ -1 & \frac{1}{2} & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

**Cramer's Rule.** If $A$ is an $n \times n$ matrix such that $\det A \neq 0$, then the system $A\vec{x} = \vec{b}$ has the unique solution

$$x_1 = \frac{\det A_1}{\det A}, \qquad x_2 = \frac{\det A_2}{\det A}, \qquad \ldots, \qquad x_n = \frac{\det A_n}{\det A},$$

where $A_j$ is obtained by replacing column $j$ of $A$ with the vector $\vec{b}$.

**Example 8.** Use Cramer's Rule to solve the linear system:

$$\begin{aligned} x_1 \qquad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

$C_3 \to C_3 - 2C_1$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \det A = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -3 & 4 & 12 \\ -1 & -2 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 12 \\ -2 & 5 \end{vmatrix} = 20 - (-24) = 44.$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, \quad \det A_1 = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = \cdots = -40.$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad \det A_2 = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = \cdots = 72.$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}, \quad \det A_3 = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = \cdots = 152.$$

The solution is:

$$x_1 = \frac{\det A_1}{\det A} = \frac{-40}{44} = -\frac{10}{11}.$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{72}{44} = \frac{18}{11}.$$

$$x_3 = \frac{\det A_3}{\det A} = \frac{152}{44} = \frac{38}{11}.$$

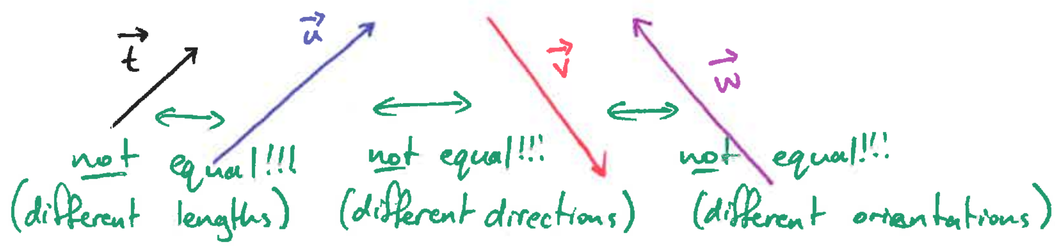
## Section 3.1: Vectors in 2-space, 3-space, and $n$-space

**Objectives.**

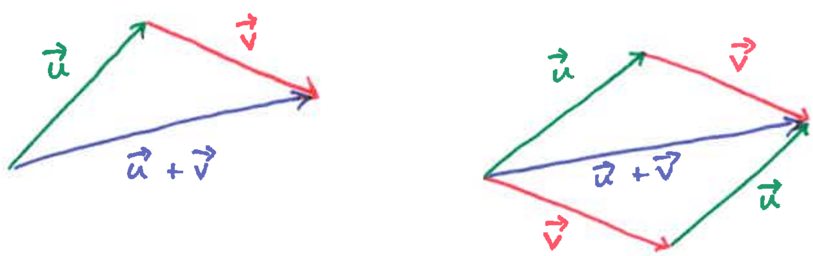
- Introduce the some terminology and notation for vectors.
- Understand vector operations in $\mathbb{R}^n$ geometrically and algebraically.
- Study some properties of vector operations.

---

A (geometric) vector is a quantity with a direction and a length, often represented by an arrow.

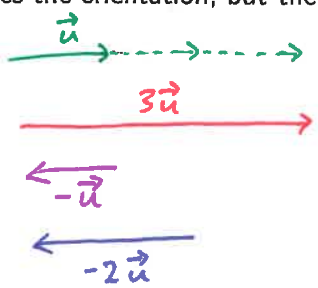



Two vectors can be added (geometrically) by placing the vectors end-to-end. (This is referred to as either the "triangle rule" or the "parallelogram rule".)




note: $\vec{u}+\vec{v}$ is the diagonal of a parallelogram with sides $\vec{u}$ and $\vec{v}$.

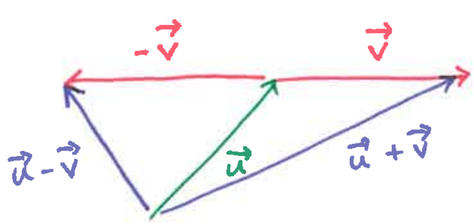
Multiplying a vector by a scalar changes ("scales") the length of the vector without changing the direction. If one vector is a scalar multiple of another, then we say the vectors are parallel. (Multiplying by a negative scalar reverses the orientation, but the result is still parallel to the original vector.)




$\vec{u}, 3\vec{u}, -\vec{u}, -2\vec{u}$ are all parallel.

note: the zero vector $\vec{0}$ is "parallel to" every vector!!!

We can view subtraction of a vector as "adding the negative of the vector".



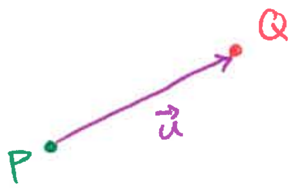
If $\underbrace{P = (a_1, a_2, \ldots, a_n)}_{\text{initial point}}$ and $\underbrace{Q = (b_1, b_2, \ldots, b_n)}_{\text{terminal point}}$ are two points in $\mathbb{R}^n$, then the vector from $P$ to $Q$ is

$$\overrightarrow{PQ} = (b_1 - a_1, \ b_2 - a_2, \ \ldots, \ b_n - a_n).$$

Two vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ are equal if their components are equal. That is:

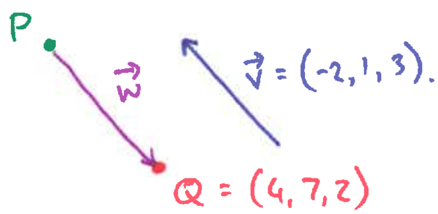
$$\vec{u} = \vec{v} \iff u_1 = v_1 \text{ and } u_2 = v_2 \text{ and } \cdots \text{ and } u_n = v_n.$$

**Example 1.** Find the vector $\vec{u} = \overrightarrow{PQ}$ that has initial point $P = (3, -1)$ and terminal point $Q = (-2, 8)$.




$$\vec{u} = \overrightarrow{PQ} = (-2-3, \ 8-(-1)) = \underline{(-5, 9)}.$$

**Example 2.** Find the initial point of a vector $\vec{w}$ that has terminal point $Q = (4, 7, 2)$ and is parallel to $\vec{v} = (-2, 1, 3)$ but has the opposite orientation. i.e. choose $\vec{w} = k\vec{v}$ where $k < 0$.




$$P = (4 + (-2), \ 7+1, \ 2+3) = \underline{(2, 8, 5)}.$$

Arithmetic with vectors (addition, subtraction, scalar multiplication) is done componentwise. If $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$ and $k$ is a scalar, then we define:

$$\vec{u} + \vec{v} = (u_1 + v_1, \ u_2 + v_2, \ \ldots, \ u_n + v_n)$$
$$k\vec{u} = (ku_1, \ ku_2, \ \ldots, \ ku_n)$$
$$-\vec{u} = (-u_1, \ -u_2, \ \ldots, \ -u_n)$$

**Example 3.** Let $\vec{u} = (3, 1, 4, -2)$ and $\vec{v} = (1, -2, 3, 0)$. Simplify:

(a) $\vec{u} + \vec{v} = (3, 1, 4, -2) + (1, -2, 3, 0) = \underline{(4, -1, 7, -2)}.$

(b) $3\vec{u} - 4\vec{v} = 3(3, 1, 4, -2) - 4(1, -2, 3, 0) = (9, 3, 12, -6) - (4, -8, 12, 0)$

$= \underline{(5, 11, 0, -6)}.$

**Properties of vector operations.** If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and $k$ and $m$ are scalars, then:

1. $(\vec{u}+\vec{v})+\vec{w} = \vec{u} + (\vec{v}+\vec{w})$
2. $\vec{u}+\vec{v} = \vec{v} + \vec{u}$
3. $\vec{u}+\vec{0} = \vec{u}$
4. $\vec{u}+(-\vec{u}) = \vec{0}$
5. $k(\vec{u}+\vec{v}) = k\vec{u} + k\vec{v}$
6. $(k+m)\vec{u} = k\vec{u} + m\vec{u}$
7. $k(m\vec{u}) = (km)\vec{u} = m(k\vec{u})$
8. $1\vec{u} = \vec{u}$

**Proof of 2.** Let $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\vec{v} = (v_1, v_2, \cdots, v_n)$. Then

$$\begin{aligned}
\vec{u}+\vec{v} &= (u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) && \text{def. of vector addition} \\
&= (u_1+v_1, u_2+v_2, \cdots, u_n+v_n) && \text{addition in } \mathbb{R} \text{ is commutative.} \\
&= (v_1+u_1, v_2+u_2, \cdots, v_n+u_n) && \text{def. of vector addition.} \\
&= (v_1, v_2, \cdots, v_n) + (u_1, u_2, \cdots, u_n) \\
&= \vec{v} + \vec{u}.
\end{aligned}$$

**Example 4.** Let $\vec{u} = (-1, 4, 6)$ and $\vec{v} = (3, 3, 3)$. Find the vector $\vec{x}$ satisfying $4\vec{x} - 2\vec{u} = 2\vec{x} - \vec{v}$.

$$4\vec{x} - 2\vec{u} = 2\vec{x} - \vec{v} \implies 2\vec{x} = 2\vec{u} - \vec{v}$$

$$\implies \vec{x} = \tfrac{1}{2}(2\vec{u} - \vec{v}) = \vec{u} - \tfrac{1}{2}\vec{v}$$

$$= (-1, 4, 6) - \tfrac{1}{2}(3,3,3)$$

$$= \left(-\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{9}{2}\right).$$

zero scalar

zero vector

**Theorem.** If $\vec{v}$ is a vector in $\mathbb{R}^n$ and $k$ is a scalar, then

1. $0\vec{v} = \vec{0}$
2. $k\vec{0} = \vec{0}$
3. $(-1)\vec{v} = -\vec{v}$

**Proof of 1.** Let $\vec{v} = (v_1, v_2, \cdots, v_n)$. Then:

$$0\vec{v} = 0(v_1, v_2, \cdots, v_n) = (0v_1, 0v_2, \cdots, 0v_n) = (0, 0, \cdots, 0) = \vec{0}.$$

A vector $\vec{w}$ in $\mathbb{R}^n$ is a <u>linear combination</u> of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_r} \in \mathbb{R}^n$ if

$$\vec{w} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + \cdots + k_r \vec{v}_r \quad \text{, where } k_1, k_2, \dots, k_r \text{ are scalars.}$$

**Example 5.** Find scalars $c_1, c_2, c_3$ satisfying $c_1(1,2,2) + c_2(0,1,-1) + c_3(3,1,2) = (-1,7,7)$.

- i.e. write $(-1,7,7)$ as a linear combination of $(1,2,2), (0,1,-1), (3,1,2)$.

This equation is equivalent to the linear system

$$\begin{aligned} c_1 \phantom{{}+c_2} + 3c_3 &= -1 \\ 2c_1 + c_2 + c_3 &= 7 \\ 2c_1 - c_2 + 2c_3 &= 7 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 2 & 1 & 1 & 7 \\ 2 & -1 & 2 & 7 \end{array}\right].$$

We can reduce this to rref using Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]. \quad \leftarrow \text{rref for linear system.}$$

That is, $c_1 = 5$, $c_2 = -1$, $c_3 = -2$.

**Example 6.** Show that there is no choice of scalars $a$ and $b$ such that $a(3,-6) + b(-1,2) = (1,1)$.

We need to solve the system

$$\begin{aligned} 3a - b &= 1 \\ -6a + 2b &= 1. \end{aligned} \longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 1 \\ -6 & 2 & 1 \end{array}\right] \xrightarrow{R_2 + 2R_3} \left[\begin{array}{cc|c} 3 & -1 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

system is inconsistent.

There is no solution!!!

## Section 3.2: Norm, Dot Product, and Distance in $\mathbb{R}^n$

**Objectives.**

- Define and apply the notions of norm and distance in $\mathbb{R}^n$.
- Introduce the dot product of two vectors, and interpret the dot product geometrically.
- Study some properties and applications of the dot product.

---

The norm (length, magnitude) of a vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

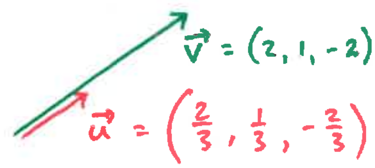
note: this generalizes Pythagoras!!!

Dividing a (non-zero) vector $\vec{v}$ by its norm produces the unit vector in the same direction as $\vec{v}$.

**Example 1.** Find the unit vector $\vec{u}$ that has the same direction as $\vec{v} = (2, 1, -2)$. Check that $\|\vec{u}\| = 1$.

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3.$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}(2, 1, -2) = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right).$$




check: $\|\vec{u}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{9}{9}} = 1.$

The distance between two points $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

**Example 2.** Find the distance between the points $\vec{u} = (1, 3, -2, 0, 2)$ and $\vec{v} = (3, 0, 1, 1, -1)$ in $\mathbb{R}^5$.

$$d(\vec{u}, \vec{v}) = \sqrt{(1-3)^2 + (3-0)^2 + (-2-1)^2 + (0-1)^2 + (2-(-1))^2}$$

$$= \sqrt{4 + 9 + 9 + 1 + 9}$$

$$= \sqrt{32}$$

$$= 4\sqrt{2}.$$

Euclidean inner product. n-dimensional Euclidean space.

The dot product of two vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

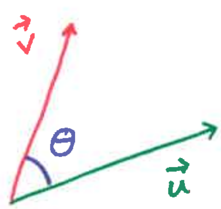
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

note: vector $\cdot$ vector = scalar.

**Example 3.** Find the dot product of the vectors $\vec{u} = (1,3,2,4)$ and $\vec{v} = (-1,1,-2,1)$

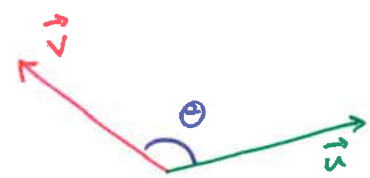
$$\vec{u} \cdot \vec{v} = (1,3,2,4) \cdot (-1,1,-2,1)$$
$$= -1 + 3 - 4 + 4$$
$$= \underline{2}.$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the dot product of two vectors is related to the angle between them. (This can also be generalized to finding "angles" between vectors in higher-dimensional spaces.)




$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

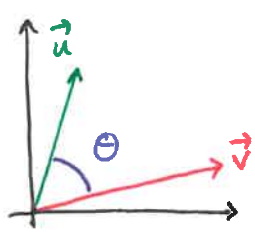
$$\Rightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$




$\theta$ acute $\iff \vec{u} \cdot \vec{v} > 0.$

$\theta$ obtuse $\iff \vec{u} \cdot \vec{v} < 0.$

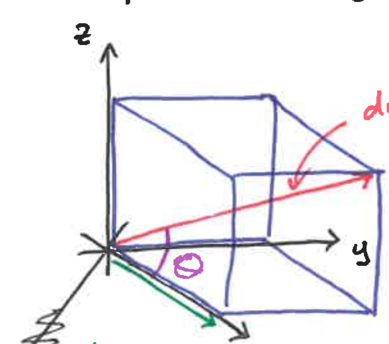
**Example 4.** Find the angle between the vectors $\vec{u} = (1,2)$ and $\vec{v} = (3,1)$.




$$\vec{u} \cdot \vec{v} = 5, \quad \|\vec{u}\| = \sqrt{5}, \quad \|\vec{v}\| = \sqrt{10}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}} \Rightarrow \underline{\theta = 45^\circ}.$$

**Example 5.** Find the angle between a diagonal and an edge of a cube.




diagonal = (1,1,1)

edge = (1,0,0)

$$\cos\theta = \frac{(1,1,1) \cdot (1,0,0)}{\|(1,1,1)\| \, \|(1,0,0)\|}$$
$$= \frac{1}{\sqrt{3}}$$
$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \underline{54.74^\circ}.$$

Notice that the dot product of a vector with itself is the square of the norm of the vector.

If $\vec{v} = (v_1, v_2, \cdots, v_n)$ is a vector in $\mathbb{R}^n$, then

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\vec{v}\|^2.$$

**Properties of the dot product.** If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and $k$ is a scalar, then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ "symmetry" (dot product commutes)
   zero vector
2. $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0 \leftarrow$ zero scalar
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
4. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

   dot product distributes over addition
5. $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$ "homogeneity"
6. $\vec{v} \cdot \vec{v} \geq 0$, and $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = \vec{0}$. "positivity"

**Example 6.** Use properties 1 and 3 above to prove property 4.

$$\begin{aligned}
(\vec{u} + \vec{v}) \cdot \vec{w} &= \vec{w} \cdot (\vec{u} + \vec{v}) && \text{by property 1} \\
&= \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v} && \text{by property 3} \\
&= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} && \text{by property 14.}
\end{aligned}$$

**Example 7.** Expand and simplify the vector expression.

$$\begin{aligned}
(2\vec{u} + 3\vec{v}) \cdot (3\vec{u} - \vec{v}) &= 2\vec{u} \cdot (3\vec{u} - \vec{v}) + 3\vec{v} \cdot (3\vec{u} - \vec{v}) \\
&= 6(\vec{u} \cdot \vec{u}) - 2(\vec{u} \cdot \vec{v}) + 9(\vec{v} \cdot \vec{u}) - 3(\vec{v} \cdot \vec{v}) \\
&= 6\|\vec{u}\|^2 - 2(\vec{u} \cdot \vec{v}) + 9(\vec{u} \cdot \vec{v}) - 3\|\vec{v}\|^2 \\
&= \underline{6\|\vec{u}\|^2 + 7(\vec{u} \cdot \vec{v}) - 3\|\vec{v}\|^2}.
\end{aligned}$$

There are two important inequalities involving norms and distances in $\mathbb{R}^n$.

**Cauchy-Schwarz Inequality.** If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

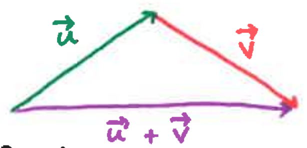
$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|.$$

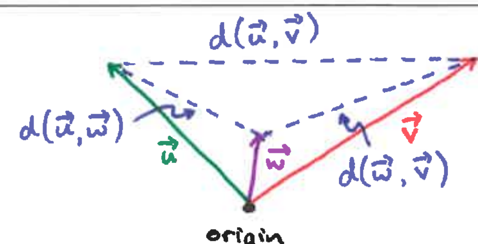
note: this implies that $-1 \leq \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} \leq 1$, so we can define the angle between $\vec{u}$ and $\vec{v}$ as $\theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right)$.

**Triangle Inequality.** If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, then:

(a) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ triangle inequality for vectors

(b) $d(\vec{u}, \vec{v}) \leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v})$ triangle inequality for distances




**Proof of (a).**

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \quad \leftarrow \text{because } \|\vec{a}\|^2 = \vec{a} \cdot \vec{a} \\
&= (\vec{u} \cdot \vec{u}) + 2(\vec{u} \cdot \vec{v}) + (\vec{v} \cdot \vec{v}) \quad \text{apply absolute value to } \vec{u} \cdot \vec{v} \\
&\leq \|\vec{u}\|^2 + 2|\vec{u} \cdot \vec{v}| + \|\vec{v}\|^2 \\
&\leq \|\vec{u}\|^2 + 2\|\vec{u}\| \, \|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2.
\end{aligned}$$

Because $\|\vec{u} + \vec{v}\| \geq 0$ and $\|\vec{u}\| + \|\vec{v}\| \geq 0$, we have $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

**Example 8.** Suppose that $\|\vec{u}\| = 4$ and $\|\vec{v}\| = 3$. What are the smallest and largest possible values of $\|\vec{u} + \vec{v}\|$?

$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| = 4 + 3 = 7.$

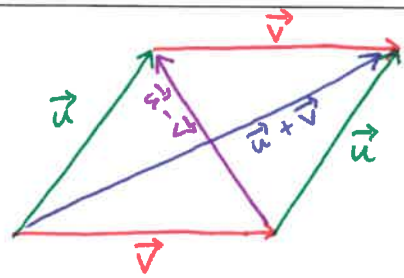
$\|\vec{u}\| = \|(\vec{u} + \vec{v}) - \vec{v}\| \leq \|\vec{u} + \vec{v}\| + \|\vec{v}\|$, so $4 \leq \|\vec{u} + \vec{v}\| + 3.$

Thus $\|\vec{u} + \vec{v}\| \geq 1$, and therefore $1 \leq \|\vec{u} + \vec{v}\| \leq 7$.

In plane geometry (that is, in $\mathbb{R}^2$), the sum of the squares of the two diagonals of a parallelogram equals the sum of the squares of the four sides. This result is also true more generally in $\mathbb{R}^n$.

**Parallelogram equation for vectors.** If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).$$




**Proof.**

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\ &= (\vec{u}\cdot\vec{u}) + 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v}) + (\vec{u}\cdot\vec{u}) - 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v}) \\ &= 2(\vec{u}\cdot\vec{u}) + 2(\vec{v}\cdot\vec{v}) \\ &= 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).\end{aligned}$$

Taking the difference of the squares of the two diagonals of a parallelogram instead gives a different expression for the dot product of two vectors.

**Theorem.** If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then:

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2.$$

**Proof.**

$$\begin{aligned}\tfrac{1}{4}\|\vec{u}+\vec{v}\|^2 - \tfrac{1}{4}\|\vec{u}-\vec{v}\|^2 &= \tfrac{1}{4}(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) - \tfrac{1}{4}(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\ &= \tfrac{1}{4}\left((\vec{u}\cdot\vec{u}) + 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v})\right) - \tfrac{1}{4}\left((\vec{u}\cdot\vec{u}) - 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v})\right) \\ &= \tfrac{1}{4}\left(4(\vec{u}\cdot\vec{v})\right) \\ &= \vec{u}\cdot\vec{v}.\end{aligned}$$

## Section 3.3: Orthogonality

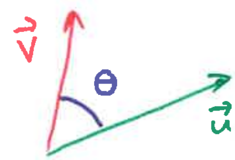
**Objectives.**

- Introduce the definition of orthogonality in $\mathbb{R}^n$.
- Represent lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$ using vector equations.
- Project a vector onto a line.
- Write a vector as the sum of two orthogonal components.

---

In Section 3.2, we defined the angle $\theta$ between two vectors $\vec{u}$ and $\vec{v}$ as

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$




The vectors $\vec{u}$ and $\vec{v}$ are orthogonal (or perpendicular) if

$$\vec{u}\cdot\vec{v} = 0.$$

note:
$\vec{u}\cdot\vec{v} > 0 \implies \theta$ is acute
$\vec{u}\cdot\vec{v} = 0 \implies \theta$ is a right angle
$\vec{u}\cdot\vec{v} < 0 \implies \theta$ is obtuse

**Example 1.** Show that the vectors $\vec{u} = (1, -2, 2, 5)$ and $\vec{v} = (3, 2, 3, -1)$ in $\mathbb{R}^4$ are orthogonal.

$$\vec{u}\cdot\vec{v} = (1,-2,2,5)\cdot(3,2,3,-1) = 3 - 4 + 6 - 5 = 0.$$

Thus $\vec{u}$ and $\vec{v}$ are orthogonal.

Notice that in $\mathbb{R}^n$, the standard basis vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are all orthogonal.

eg. $\vec{e}_1 \cdot \vec{e}_n = (1,0,\cdots,0)\cdot(0,0,\ldots,1) = 0.$

---

**Pythagorean Theorem in $\mathbb{R}^n$.** If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ then

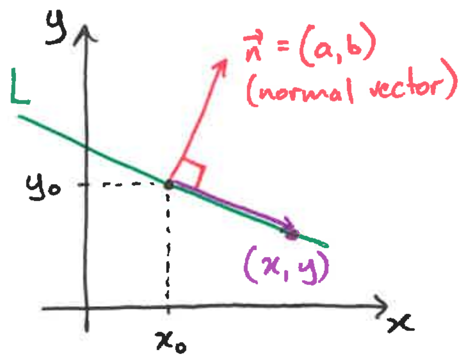
$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

---

**Proof.** $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = (\vec{u}\cdot\vec{u}) + 2\underbrace{(\vec{u}\cdot\vec{v})}_{=0} + (\vec{v}\cdot\vec{v})$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2.$

A straight line in $\mathbb{R}^2$ can be described by specifying a point and a <u>normal</u> direction (that is, a vector orthogonal to the line).




If $(x,y)$ is <u>any</u> point on the line $L$, then $(x-x_0, y-y_0)$ is orthogonal to $\vec{n}$.

$$\vec{n} \cdot (x-x_0, y-y_0) = 0$$

$$(a,b) \cdot (x-x_0, y-y_0) = 0$$

$$\boxed{a(x-x_0) + b(y-y_0) = 0}$$

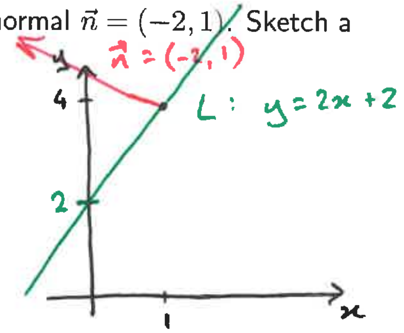
<u>or</u>: $ax + by + c = 0.$

**Example 2.** Write an equation for the line in $\mathbb{R}^2$ through the point $(1,4)$ with normal $\vec{n} = (-2,1)$. Sketch a diagram indicating the point, the normal vector, and the line.

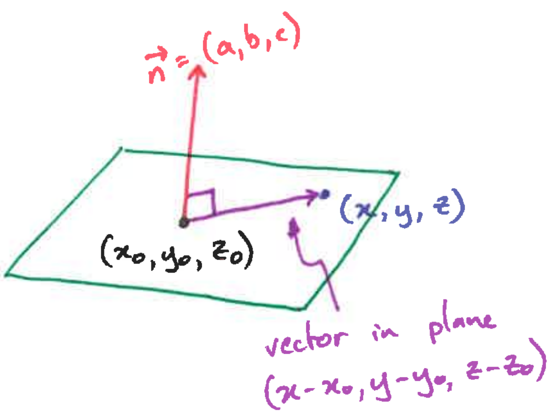
$$\vec{n} \cdot (x-x_0, y-y_0) = 0 \Rightarrow (-2,1) \cdot (x-1, y-4) = 0$$

$$\Rightarrow -2(x-1) + 1(y-4) = 0$$

$$\Rightarrow \underline{-2x + y - 2 = 0.}$$




The same idea can be used to write equations for planes in $\mathbb{R}^3$.




$$\vec{n} \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$(a,b,c) \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

<u>or</u>: $ax + by + cz + d = 0$

**Example 3.** Write an equation for the plane in $\mathbb{R}^3$ through the point $(2,-5,0)$ with normal $\vec{n} = (1,3,-1)$.

$$\vec{n} \cdot (x-x_0, y-y_0, z-z_0) = 0 \Rightarrow (1,3,-1) \cdot (x-2, y+5, z) = 0$$

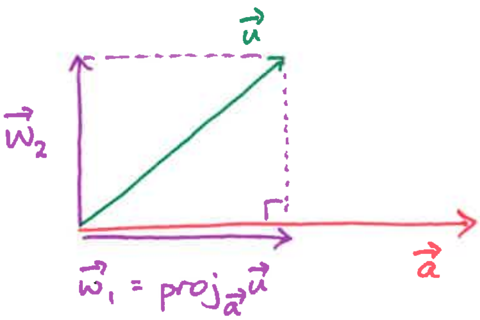
$$\Rightarrow (x-2) + 3(y+5) - z = 0$$

$$\Rightarrow \underline{x + 3y - z + 13 = 0.}$$

In Chapter 1, we introduced (orthogonal) projections onto the coordinate axes as examples or linear transformations. We can now extend this idea to (orthogonal) projections onto any line in $\mathbb{R}^n$.

**Projection Theorem.** If $\vec{u}$ and $\vec{a}$ are vectors in $\mathbb{R}^n$ with $\vec{a} \neq \vec{0}$, then $\vec{u}$ can be written in exactly one way as $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\vec{w}_1$ is parallel to $\vec{a}$ and $\vec{w}_2$ is orthogonal to $\vec{a}$. Specifically:

$$\vec{w}_1 = \text{proj}_{\vec{a}}\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\,\vec{a} \qquad \text{and} \qquad \vec{w}_2 = \vec{u} - \text{proj}_{\vec{a}}\vec{u} = \vec{u} - \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\,\vec{a}.$$




why? $\vec{w}_1 = k\vec{a}$ and $\vec{w}_2 \cdot \vec{a} = 0$, so

$$\vec{u}\cdot\vec{a} = (\vec{w}_1 + \vec{w}_2)\cdot\vec{a} = \vec{w}_1\cdot\vec{a} + \vec{w}_2\cdot\vec{a}$$

$$= k\vec{a}\cdot\vec{a} + 0 = k\|\vec{a}\|^2.$$

$$\Longrightarrow \quad k = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}, \text{ so}$$

$$\vec{w}_1 = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}, \quad \vec{w}_2 = \vec{u} - \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}.$$

**Example 4.** Let $\vec{u} = (1, 2, 3)$ and $\vec{a} = (4, -1, -1)$. Find the component of $\vec{u}$ parallel to $\vec{a}$ and the component of $\vec{u}$ orthogonal to $\vec{a}$.

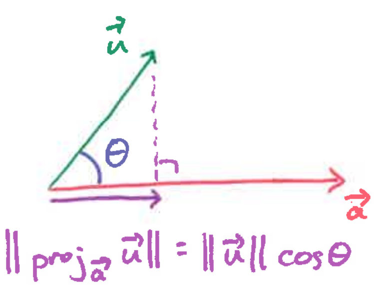
component $\parallel$ to $\vec{a}$:

$$\text{proj}_{\vec{a}}\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a} = \frac{(1,2,3)\cdot(4,-1,-1)}{(4,-1,-1)\cdot(4,-1,-1)}(4,-1,-1) = \frac{-1}{18}(4,-1,-1) = \left(-\frac{2}{9}, \frac{1}{18}, \frac{1}{18}\right).$$

component $\perp$ to $\vec{a}$:

$$\vec{u} - \text{proj}_{\vec{a}}\vec{u} = (1,2,3) - \left(-\frac{2}{9}, \frac{1}{18}, \frac{1}{18}\right) = \left(\frac{11}{9}, \frac{35}{18}, \frac{53}{18}\right).$$

The norm of the orthogonal projection (of $\vec{u}$ onto $\vec{a}$ can be written either in terms of the two vectors or in terms of $\vec{u}$ and the angle $\theta$ between $\vec{u}$ and $\vec{a}$.



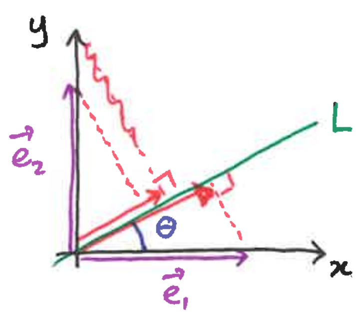

$$\|\text{proj}_{\vec{a}}\vec{u}\| = \left\|\frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}\right\| = \frac{|\vec{u}\cdot\vec{a}|}{\|\vec{a}\|^2}\|\vec{a}\| = \frac{|\vec{u}\cdot\vec{a}|}{\|\vec{a}\|}$$

$$= \frac{\|\vec{u}\|\|\vec{a}\|\cos\theta}{\|\vec{a}\|} = \|\vec{u}\|\cos\theta.$$

assuming $\theta$ is acute!!!

**Example 5.** Let $L$ be a line through the origin in $\mathbb{R}^2$ that makes an angle $\theta$ with the positive $x$-axis.

(a) Find the projections of $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$ onto $L$.




$\vec{a} = (\cos\theta, \sin\theta)$ is a vector in the direction of $L$.

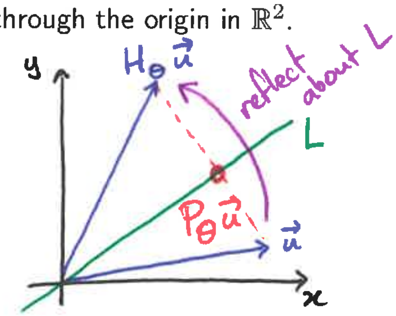
$$\text{proj}_{\vec{a}}\,\vec{e}_1 = \frac{(1,0)\cdot(\cos\theta,\sin\theta)}{1^2}(\cos\theta,\sin\theta) = (\cos^2\theta, \cos\theta\sin\theta)$$

$$\text{proj}_{\vec{a}}\,\vec{e}_2 = \frac{(0,1)\cdot(\cos\theta,\sin\theta)}{1^2}(\cos\theta,\sin\theta) = (\cos\theta\sin\theta, \sin^2\theta).$$

(b) Find the standard matrix $P_\theta$ for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that projects each point onto $L$.

$$P_\theta = \left[\ \text{proj}_{\vec{a}}\,\vec{e}_1 \ \middle|\ \text{proj}_{\vec{a}}\,\vec{e}_2\ \right] = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.$$

We can use the previous example to find a linear transformation that reflects a vector/point about a line through the origin in $\mathbb{R}^2$.




$$P_\theta\vec{u} = \frac{1}{2}\left(H_\theta\vec{u} + \vec{u}\right)$$

$$\Rightarrow \quad P_\theta = \frac{1}{2}\left(H_\theta + I\right)$$

$$\Rightarrow \quad H_\theta = 2P_\theta - I = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

**Example 6.** Let $\vec{x} = (4,1)$ and let $L$ be the line through the origin that makes an angle of $\pi/3$ with the positive $x$-axis.

(a) Find the projection of $\vec{x}$ onto $L$.

$$P_{\frac{\pi}{3}} = \begin{bmatrix} \cos^2\frac{\pi}{3} & \cos\frac{\pi}{3}\sin\frac{\pi}{3} \\ \cos\frac{\pi}{3}\sin\frac{\pi}{3} & \sin^2\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}, \quad \text{so} \quad P_{\frac{\pi}{3}}(4,1) = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\sqrt{3}}{4} \\ \sqrt{3} + \frac{3}{4} \end{bmatrix}$$

(b) Find the reflection of $\vec{x}$ about $L$.

$$H_{\frac{\pi}{3}} = \begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \text{so} \quad H_{\frac{\pi}{3}}(4,1) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + \frac{\sqrt{3}}{2} \\ 2\sqrt{3} + \frac{1}{2} \end{bmatrix}$$

**Distance problems.**

The distance between a point and a line in $\mathbb{R}^2$ or between a point and a plane in $\mathbb{R}^3$ can be found using projections.

**Theorem.**

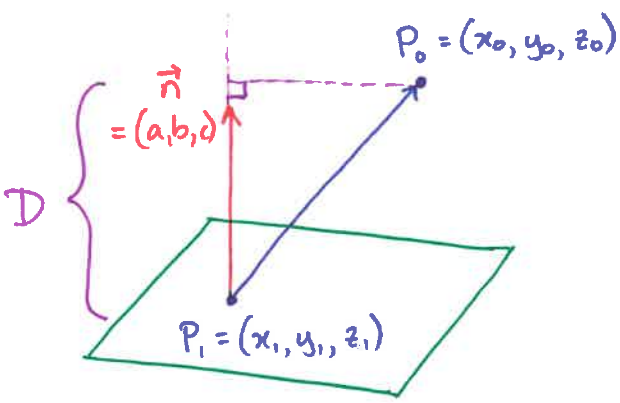
1. In $\mathbb{R}^2$, the distance between the point $P_0 = (x_0, y_0)$ and the line $ax + by + c = 0$ is
$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

normal is $\vec{n} = (a,b,c)$

2. In $\mathbb{R}^3$, the distance between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is
$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

**Proof of 2.**

Choose $P_1 = (x_1, y_1, z_1)$ in the plane, and project $\overrightarrow{P_1P_0}$ onto $\vec{n}$.




$$D = \|\text{proj}_{\vec{n}} \overrightarrow{P_1P_0}\|$$
$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a,b,c)|}{\|(a,b,c)\|}$$
$$= \frac{|ax_0 - ax_1 + by_0 - by_1 + cz_0 - cz_1|}{\sqrt{a^2+b^2+c^2}}$$
$$= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}.$$

b/c $P_1$ is in the plane,

$ax_1 + by_1 + cz_1 + d = 0.$

$\Rightarrow -ax_1 - by_1 - cz_1 = d.$

**Example 7.** Find the distance in $\mathbb{R}^2$ between the point $(1, -1)$ and the line $x + 2y = 3$.

$a = 1,\ b = 2,\ c = -3$

$$D = \frac{|1(1) + 2(-1) + (-3)|}{\sqrt{1^2 + 2^2}} = \frac{|-4|}{\sqrt{5}} = \frac{4}{\sqrt{5}}.$$

## Section 3.4: The Geometry of Linear Systems

**Objectives.**

- Write vector and parametric equations for lines and planes in $\mathbb{R}^n$.
- Express a line segment in vector form.

---

In Section 3.3, we saw how the dot product allows us to write vector and scalar equations for a line in $\mathbb{R}^2$ or a plane in $\mathbb{R}^3$. Specifically:

- the line in $\mathbb{R}^2$ through the point $\vec{x}_0 = (x_0, y_0)$ and normal to the vector $\vec{n} = (a, b)$ is

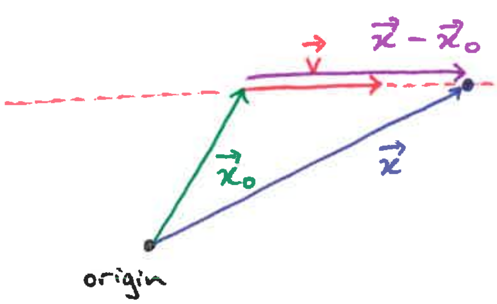
$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \quad \text{or} \quad a(x - x_0) + b(y - y_0) = 0.$$

- the plane in $\mathbb{R}^3$ through the point $\vec{x}_0 = (x_0, y_0, z_0)$ and normal to the vector $\vec{n} = (a, b, c)$ is

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \quad \text{or} \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

In this section, we will explore how the equation of a line in higher dimensions can be written using a point on the line and a direction parallel to the line, and how the equation of a plane in higher dimensions can be written using a point on the plane and two (non-parallel!) directions parallel to the plane.

Suppose that $\vec{x}$ is a general point on the line through the point $\vec{x}_0$ and parallel to the vector $\vec{v}$.




A vector on this line is a scalar multiple of $\vec{v}$.

$$\vec{x} - \vec{x}_0 = t\vec{v}$$

parameter

$$\Rightarrow \quad \vec{x} = \vec{x}_0 + t\vec{v}$$

gen. pt. = fixed pt. + parameter · direction

**Example 1.** Let $L$ be the line in $\mathbb{R}^3$ through the point $\vec{x}_0 = (3, -1, 5)$ and parallel to the vector $\vec{v} = (-2, 1, 2)$.

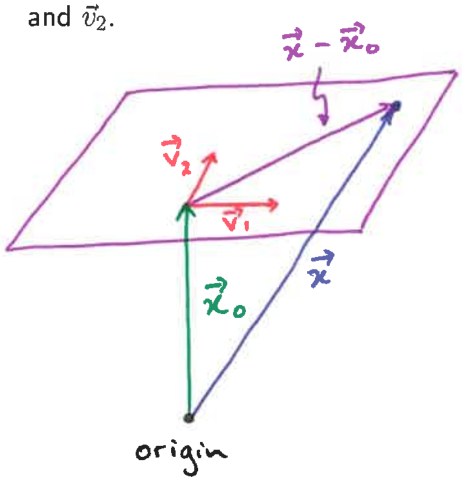
(a) Find a vector equation for the line $L$.

$$\vec{x} = \vec{x}_0 + t\vec{v} = (3, -1, 5) + t(-2, 1, 2). = (3 - 2t, -1 + t, 5 + 2t).$$

(b) Find parametric equations for the line $L$.

$$x = 3 - 2t, \quad y = -1 + t, \quad z = 5 + 2t$$

Suppose $\vec{x}$ is a general point on the plane through the point $\vec{x}_0$ and parallel to the (non-parallel) vectors $\vec{v}_1$ and $\vec{v}_2$.




A vector $\vec{x} - \vec{x}_0$ in the plane is a linear combination of $\vec{v}_1$ and $\vec{v}_2$

$$\vec{x} - \vec{x}_0 = t_1 \vec{v}_1 + t_2 \vec{v}_2$$

$$\Rightarrow \vec{x} = \vec{x}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 .$$

**Example 2.** Consider the point $\vec{x}_0 = (1, 4, 0, -3)$ in $\mathbb{R}^4$ and the vectors $\vec{v}_1 = (2, -1, 1, 0)$ and $\vec{v}_2 = (3, -6, 5, 2)$.

(a) Find a vector equation for the plane through $\vec{x}_0$ and parallel to both $\vec{v}_1$ and $\vec{v}_2$.

$$\vec{x} = \vec{x}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 = (1, 4, 0, -3) + t_1(2, -1, 1, 0) + t_2(3, -6, 5, 2).$$

(b) Find parametric equations for the plane in part (a).

$$w = 1 + 2t_1 + 3t_2, \quad x = 4 - t_1 - 6t_2, \quad y = t_1 + 5t_2, \quad z = -3 + 2t_2.$$

**Example 3.** The scalar equation $x + 2y + 3z = 4$ represents a plane in $\mathbb{R}^3$.

(a) Find parametric equations for the plane. $\longrightarrow$ use two variables as parameters!!!

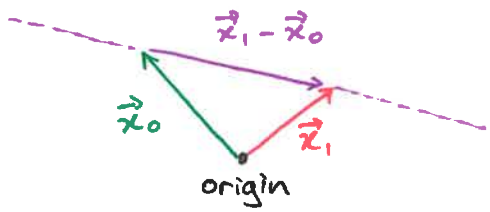
Let $y = t_1$ and $z = t_2$. Then

$$x = 4 - 2t_1 - 3t_2 .$$

(b) Find a vector equation for the plane.

$$\vec{x} = (4 - 2t_1 - 3t_2, t_1, t_2) = (4, 0, 0) + t_1(-2, 1, 0) + t_2(-3, 0, 1).$$

Any two distinct points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^n$ determine a unique line:




$$\vec{x} = \vec{x}_0 + t\,\underline{(\vec{x}_1 - \vec{x}_0)}$$

i.e. $\vec{v} = \vec{x}_1 - \vec{x}_0$ is a direction parallel to the line.

or

$$\vec{x} = (1-t)\,\vec{x}_0 + t\,\vec{x}_1$$

**Example 4.** Consider the two points $\vec{x}_0 = (1, -1)$ and $\vec{x}_1 = (0, 3)$ in $\mathbb{R}^2$.

(a) Find a vector equation for the line through $\vec{x}_0$ and $\vec{x}_1$.

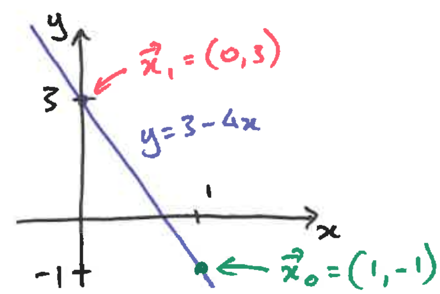
$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) = (1,-1) + t\big(0-1,\ 3-(-1)\big) = (1,-1) + t(-1,4).$$

(b) Write a scalar equation for the line in part (a).

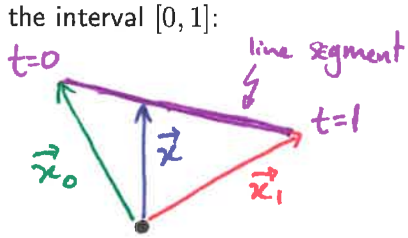
From $x = 1-t$ and $y = -1+4t$, we have

$t = 1-x$ and $4t = 1+y$.

Thus $4(1-x) = 1+y$, or $\underline{y = 3-4x}$.




To describe the line segment connecting two points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^n$, we can restrict the values of the parameter $t$ to the interval $[0, 1]$:




$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0), \quad 0 \le t \le 1$$

or

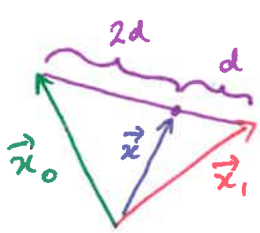
$$\vec{x} = (1-t)\vec{x}_0 + t\vec{x}_1, \quad 0 \le t \le 1$$

**Example 5.** Consider the two points $\vec{x}_0 = (1, -4, -2, 5)$ and $\vec{x}_1 = (4, -2, 7, 2)$.

(a) Find an equation for the line segment from $\vec{x}_0$ to $\vec{x}_1$.

$$\vec{x} = (1-t)(1,-4,-2,5) + t(4,-2,7,2), \quad 0 \le t \le 1.$$

(b) Find the point on this line segment for which the distance to $\vec{x}_0$ is twice the distance to $\vec{x}_1$.




- use $t = \frac{2}{3}$ (i.e. $\frac{2}{3}$ of distance from $\vec{x}_0$ to $\vec{x}_1$)

$$\vec{x} = \left(1-\tfrac{2}{3}\right)(1,-4,-2,5) + \tfrac{2}{3}(4,-2,7,2)$$

$$= \left(\tfrac{1}{3}, -\tfrac{4}{3}, -\tfrac{2}{3}, \tfrac{5}{3}\right) + \left(\tfrac{8}{3}, -\tfrac{4}{3}, \tfrac{14}{3}, \tfrac{4}{3}\right)$$

$$= \left(3, -\tfrac{8}{3}, 4, 3\right).$$

Recall that a homogeneous linear equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

or: $\vec{a} \cdot \vec{x} = 0$, where $\vec{a} = (a_1, a_2, \dots, a_n)$ and $\vec{x} = (x_1, x_2, \dots, x_n)$.

Notice from this that every vector that satisfies a homogeneous linear equation is orthogonal to the coefficient vector. In particular, any solution to the matrix equation $A\vec{x} = \vec{0}$ is orthogonal to every row of the matrix $A$.

**Theorem.** If $A$ is an $m \times n$ matrix, then the set of solutions to the homogeneous linear system $A\vec{x} = \vec{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row of $A$.

**Example 6.** The linear system

$$\begin{bmatrix} 1 & 5 & -10 & 0 & 2 \\ 3 & -2 & 0 & 2 & 1 \\ 4 & 2 & 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solution $x_1 = -2t$, $x_2 = 2s$, $x_3 = s + t$, $x_4 = 2s$, $x_5 = 6t$. Show that the vector

$$\vec{x} = (-2t, 2s, s+t, 2s, 6t) \longleftarrow \text{solutions for system!!!}$$

is orthogonal to every row of the coefficient matrix for the system.

$$\vec{r}_1 \cdot \vec{x} = (1, 5, -10, 0, 2) \cdot (-2t, 2s, s+t, 2s, 6t)$$
$$= -2t + 10s - 10(s+t) + 0(2s) + 2(6t)$$
$$= -2t + 10s - 10s - 10t + 12t = \underline{0.}$$

$$\vec{r}_2 \cdot \vec{x} = (3, -2, 0, 2, 1) \cdot (-2t, 2s, s+t, 2s, 6t)$$
$$= 3(-2t) - 2(2s) + 0(s+t) + 2(2s) + 1(6t)$$
$$= -6t - 4s + 4s + 6t = \underline{0.}$$

$$\vec{r}_3 \cdot \vec{x} = (4, 2, 2, -3, 1) \cdot (-2t, 2s, s+t, 2s, 6t)$$
$$= -8t + 4s + 2s + 2t - 6s + 6t = \underline{0.}$$

## Section 3.5: Cross Product

**Objectives.**

- Introduce the cross product of two vectors in $\mathbb{R}^3$.
- Interpret the cross product geometrically.
- Study some properties of the cross product.

---

The cross product of two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is

note: vector x vector = vector.

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

$$= \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

(Note that the cross product is only defined for vectors in $\mathbb{R}^3$.)

**Example 1.** Compute $\vec{u} \times \vec{v}$ for the vectors $\vec{u} = (2, 3, -2)$ and $\vec{v} = (1, 4, 1)$.

$$\vec{u} \times \vec{v} = \big((3)(1) - (-2)(4),\ (-2)(1) - (2)(1),\ (2)(4) - (3)(1)\big)$$

$$= (11, -4, 5).$$

The cross product can also be expressed as a $3 \times 3$ determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

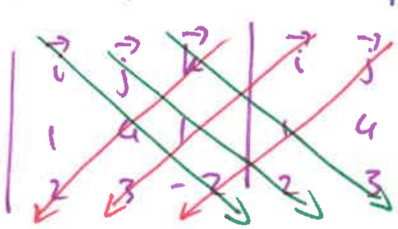
**Example 2.** Compute $\vec{v} \times \vec{u}$ for the vectors in Example 1. What do you notice?

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

$$\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 1 \\ 2 & 3 & -2 \end{vmatrix} = -8\vec{i} + 2\vec{j} + 3\vec{k} - 8\vec{k} - 3\vec{i} + 2\vec{j}$$

$$= -11\vec{i} + 4\vec{j} - 5\vec{k}$$

$$= (-11, 4, -5).$$



**Properties of the cross product.** If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$ and $k$ is a scalar, then:

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ anticommutative
2. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ cross products distributes over addition
3. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
4. $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$ scalar multiples behave "nicely"
5. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{u} = \vec{0}$

**Proof of 1.** Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -(\vec{v} \times \vec{u}).$$

**Example 3.** Show that $(\vec{u} + k\vec{v}) \times \vec{v} = \vec{u} \times \vec{v}$.

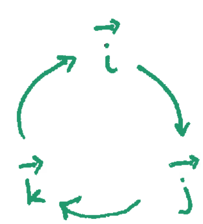
$$(\vec{u} + k\vec{v}) \times \vec{v} = (\vec{u} \times \vec{v}) + (k\vec{v} \times \vec{v}) = (\vec{u} \times \vec{v}) + k(\cancelto{\vec{0}}{\vec{v} \times \vec{v}})$$
$$= (\vec{u} \times \vec{v}) + k\vec{0} = \vec{u} \times \vec{v}.$$

**Example 4.** Compute the following cross products, where $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$.

(a) $\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$

(b) $\vec{j} \times \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}$

(c) $\vec{k} \times \vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 0 & 0 \\ 1 & 0 \end{matrix} = \vec{j}$



$\vec{i} \times \vec{i} = \vec{0}$

$\vec{i} \times \vec{j} = \vec{k}$

$\vec{i} \times \vec{k} = -\vec{j}$

An important property of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

**Relationships between the dot product and the cross product.** If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$, then:

1. $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ i.e. $\vec{u}$ is orthogonal to $\vec{u} \times \vec{v}$
2. $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \, \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$ Lagrange's identity
3. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ scalar triple product
4. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ vector triple product

**Proof of 1.** Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Then:

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = (u_1, u_2, u_3) \cdot (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

$$= \underline{u_1 u_2 v_3} - \underline{u_1 u_3 v_2} + \underline{u_2 u_3 v_1} - \underline{u_2 u_1 v_3} + \underline{u_3 u_1 v_2} - \underline{u_3 u_2 v_1}$$

$$= 0.$$

Therefore $\vec{u}$ and $\vec{u} \times \vec{v}$ are orthogonal.

**Example 5.** For the vectors $\vec{u} = (2, 3, -2)$ and $\vec{v} = (1, 4, 1)$ in Example 1, confirm that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

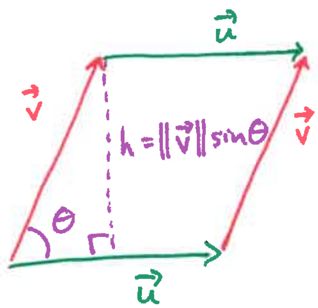
<u>recall</u>: $\vec{u} \times \vec{v} = (11, -4, 5)$.

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = (2, 3, -2) \cdot (11, -4, 5) = 22 - 12 - 10 = 0$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = (1, 4, 1) \cdot (11, -4, 5) = 11 - 16 + 5 = 0$$

Thus both $\vec{u}$ and $\vec{v}$ are orthogonal to $\vec{u} \times \vec{v}$.

The norm of $\vec{u} \times \vec{v}$ is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.




(from Lagrange:) $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

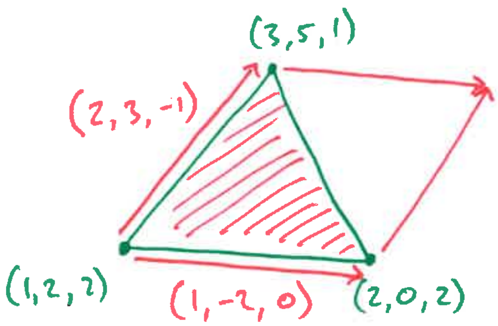
$= \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2\theta$

$= \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2\theta)$

$= \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta$

Therefore: $\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta}$ ← area of parallelogram.

**Example 6.** Find the area of the triangle with vertices $(1, 2, 2)$, $(3, 5, 1)$, and $(2, 0, 2)$.

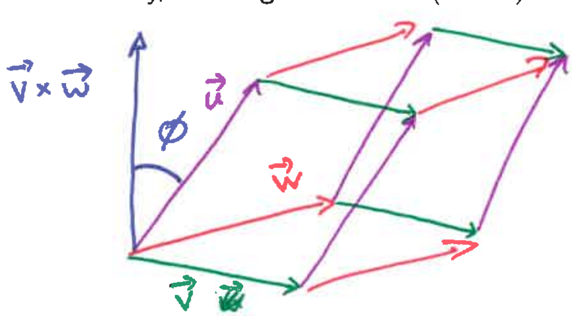



area $= \frac{1}{2} \|(1,-2,0) \times (2,3,-1)\|$

$= \frac{1}{2} \|(2,1,7)\|$

$= \underline{\frac{1}{2}\sqrt{54}}$.

Similarly, the magnitude of $\vec{u} \cdot (\vec{v} \times \vec{w})$ is the volume of the parallelipiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$.




volume = height × area of base

$= \left(\|\vec{u}\| |\cos\phi|\right)\left(\|\vec{v} \times \vec{w}\|\right)$

$= \|\vec{u}\| \|\vec{v} \times \vec{w}\| |\cos\phi|$

$= |\vec{u} \cdot (\vec{v} \times \vec{w})|$

**Example 7.** Find the volume of the parallelipiped spanned by $(1, 2, 2)$, $(3, 5, 1)$, and $(2, 0, 2)$.

volume $= |(1,2,2) \cdot ((3,5,1) \times (2,0,2))| = |(1,2,2) \cdot (10,-4,-10)|$

$= |10 - 8 - 20| = |-18| = \underline{18}$.

**Theorem.** The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ lie in the same plane if and only if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

i.e. the volume spanned by $\vec{u}, \vec{v}, \vec{w}$ is zero, so these vectors determine a flat surface rather than a parallelepiped.

**Section 3.3: Orthogonal projections in $\mathbb{R}^3$**

The orthogonal projections of a vector $\vec{x} = (x, y, z)$ in $\mathbb{R}^3$ onto each of the coordinate axes are given by:

$$\begin{aligned} T_x(\vec{x}) &= (x, 0, 0) && \textit{projection onto } x\textit{-axis}, \\ T_y(\vec{x}) &= (0, y, 0) && \textit{projection onto } y\textit{-axis}, \\ T_z(\vec{x}) &= (0, 0, z) && \textit{projection onto } z\textit{-axis}. \end{aligned}$$

**Problem 1.** Let $\vec{x} = (x, y, z)$ be a vector in $\mathbb{R}^3$.

(a) Show that the vectors $T_x(\vec{x})$ and $T_y(\vec{x})$ are orthogonal.

$$\begin{aligned} T_x(\vec{x}) \cdot T_y(\vec{x}) &= (x,0,0)\cdot(0,y,0) \\ &= x\cdot 0 + 0\cdot y + 0\cdot 0 \\ &= 0. \end{aligned}$$

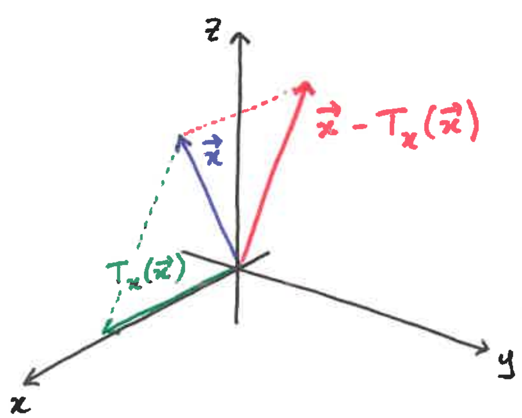
Thus $T_x(\vec{x})$ and $T_y(\vec{x})$ are orthogonal.

(b) Show that the vectors $T_x(\vec{x})$ and $\vec{x} - T_x(\vec{x})$ are orthogonal.

$$\begin{aligned} T_x(\vec{x})\cdot(\vec{x} - T_x(\vec{x})) &= (x,0,0)\cdot((x,y,z)-(x,0,0)) \\ &= (x,0,0)\cdot(0,y,z) \\ &= 0. \end{aligned}$$

Thus $T_x(\vec{x})$ and $\vec{x} - T_x(\vec{x})$ are orthogonal.

(c) Sketch a diagram showing $\vec{x}$, $T_x(\vec{x})$, and $\vec{x} - T_x(\vec{x})$.



**Section 3.4: Transformations of lines in $\mathbb{R}^n$**

Recall that a line in $\mathbb{R}^n$ can be represented by the equation

$$\vec{x} = \vec{x}_0 + t\vec{v},$$

where $\vec{x}$ is a general point on the line, $\vec{x}_0$ is a fixed point on the line, and $\vec{v}$ is a nonzero vector parallel to the line.

**Problem 2.** Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator, so that $A$ is an invertible $n \times n$ matrix.

(a) Show that the image of the line $\vec{x} = \vec{x}_0 + t\vec{v}$ in $\mathbb{R}^n$ under the transformation $T_A$ is also a line in $\mathbb{R}^n$.

$$\begin{aligned} T_A(\vec{x}) &= T_A(\vec{x}_0 + t\vec{v}) \\ &= T_A(\vec{x}_0) + t\,T_A(\vec{v}) \\ &= A\vec{x}_0 + t\,A\vec{v}. \end{aligned}$$

Because $A\vec{x}_0$ is a vector in $\mathbb{R}^n$, and $A\vec{v}$ is a nonzero vector in $\mathbb{R}^n$ (since $A$ is invertible), this represents a line in $\mathbb{R}^n$.

(b) Let $A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$. Find vector and parametric equations for the image of the line $\vec{x} = (1,3) + t(2,-1)$ under multiplication by $A$.

$$A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}, \quad A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

The image of the line $\vec{x} = (1,3) + t(2,-1)$ is the line $\vec{x} = (5,-9) + t(3,10)$.

The parametric equations are $x = 5 + 3t$ and $y = -9 + 10t$.

## Section 10.1: Constructing Curves and Surfaces Through Specified Points

### Lines in $\mathbb{R}^2$

Any two distinct points $(x_1, y_1)$, $(x_2, y_2)$ in $\mathbb{R}^2$ lie a (unique) line $c_1x + c_2y + c_3 = 0$, where at least one of $c_1$ and $c_2$ is not zero. This implies that the homogeneous linear system

$$\begin{aligned} xc_1 + \ \ yc_2 + c_3 &= 0 \\ x_1c_1 + y_1c_2 + c_3 &= 0 \\ x_2c_1 + y_2c_2 + c_3 &= 0 \end{aligned}$$

has a non-trivial solution; equivalently the determinant of the coefficient matrix is zero, which gives the following equation for the line through $(x_1, y_1)$ and $(x_2, y_2)$.

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

**Problem 3.** Consider the line in $\mathbb{R}^2$ through the two points $(3, 1)$ and $(5, -8)$.

(a) Use the determinant above to find an equation for the line.

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 5 & -8 & 1 \end{vmatrix} = 0 \Rightarrow x\begin{vmatrix} 1 & 1 \\ -8 & 1 \end{vmatrix} - y\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 5 & -8 \end{vmatrix} = 0$$

$$\Rightarrow x(1+8) - y(3-5) + (-24-5) = 0$$

$$\Rightarrow \underline{9x + 2y - 29 = 0}.$$

(b) Find the points where the line intersects each of the coordinate axes.

If $y=0$ then $x = \frac{29}{9}$. If $x=0$ then $y = \frac{29}{2}$.

The line intersects the axes at $\left(\frac{29}{9}, 0\right)$ and $\left(0, \frac{29}{2}\right)$.

(c) Graph the equation from part (a) to confirm that the line passes through the two given points.

## Circles in $\mathbb{R}^2$

The same method can be used to find a determinant equation for the unique circle

$$c_1(x^2+y^2)+c_2x+c_3y+c_4=0$$

through three points $(x_1,y_1)$, $(x_2,y_2)$, and $(x_3,y_3)$ not on the same line.

**Problem 4.** Suppose the three points $(x_1,y_1)$, $(x_2,y_2)$, and $(x_3,y_3)$ all lie on the circle $c_1(x^2+y^2)+c_2x+c_3y+c_4=0$.

(a) Set up a homogeneous system of linear equations in $c_1$, $c_2$, $c_3$, and $c_4$ satisfied by the three given points and a general point $(x,y)$ on the same circle.

$$\begin{aligned} c_1(x^2+y^2)+c_2x+c_3y+c_4&=0\\ c_1(x_1^2+y_1^2)+c_2x_1+c_3y_1+c_4&=0\\ c_1(x_2^2+y_2^2)+c_2x_2+c_3y_2+c_4&=0\\ c_1(x_3^2+y_3^2)+c_2x_3+c_3y_3+c_4&=0 \end{aligned}$$

(b) The system in part (a) has non-trivial solutions. Write a determinant equation to represent this.

$$\begin{vmatrix} x^2+y^2 & x & y & 1\\ x_1^2+y_1^2 & x_1 & y_1 & 1\\ x_2^2+y_2^2 & x_2 & y_2 & 1\\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0$$

(c) Find the center and the radius of the circle passing through $(2,-2)$, $(3,5)$, and $(-4,6)$.

$$\begin{vmatrix} x^2+y^2 & x & y & 1\\ 8 & 2 & -2 & 1\\ 34 & 3 & 5 & 1\\ 52 & -4 & +6 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad 50x^2+100x+50y^2-200y-1000=0$$

$$\Rightarrow \quad x^2+2x+y^2-4y=20$$

$$\Rightarrow \quad (x+1)^2+(y-2)^2=25 \qquad \text{center}=(-1,2),\ \text{radius}=5.$$

(d) Graph the equation from part (c) to confirm that the circle passes through the three given points.

## Conic sections in $\mathbb{R}^2$

A general conic section in $\mathbb{R}^2$ has equation

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0,$$

and is determined by five distinct points in the plane.

**Problem 5.** (a) Find a determinant equation for the conic section through the five distinct points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5).$$

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$$

(b) Find an equation for the conic section through the points $(0, 0)$, $(0, -1)$, $(2, 0)$, $(2, -5)$, and $(4, -1)$.

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 4 & 0 & 0 & 2 & 0 & 1 \\ 4 & -10 & 25 & 2 & -5 & 1 \\ 16 & -4 & 1 & 4 & -1 & 1 \end{vmatrix} = 0$$

$$\Longrightarrow \quad 160x^2 + 320xy + 320y^2 - 320x + 320y = 0$$

$$\Longrightarrow \quad x^2 + 2xy + y^2 - 2x + 2y = 0$$

(c) Graph the equation from part (b). What type of conic section is this?

a parabola!!!

## Planes in $\mathbb{R}^3$

A plane in $\mathbb{R}^3$ has the scalar equation $c_1x + c_2y + c_3z + c_4 = 0$, and is determined by three points not on the same line.

**Problem 6.** (a) Find a determinant equation for the plane through the three points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

(b) Find a scalar equation of the plane through the points $(2, 1, 3)$, $(2, -1, -1)$, and $(1, 1, 2)$.

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 3 & 1 \\ 2 & -1 & -1 & 1 \\ 1 & 1 & 2 & 1 \end{vmatrix} = 0 \implies 2x + 4y - 2z - 1 = 0.$$

(c) Graph the equation from part (c) to confirm that the plane passes through the three given points.

## Spheres in $\mathbb{R}^3$

A sphere in $\mathbb{R}^3$ has equation

$$c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0,$$

and is determined by four points not in the same plane.

**Problem 7.** (a) Find a determinant equation for the sphere through the four points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$, and $(x_4, y_4, z_4)$.

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

(b) Find an equation of the sphere through the points $(0, 1, -2)$, $(1, 3, 1)$, $(2, -1, 0)$, and $(3, 1, -1)$.

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 5 & 0 & 1 & -2 & 1 \\ 11 & 1 & 3 & 1 & 1 \\ 5 & 2 & -1 & 0 & 1 \\ 11 & 3 & 1 & -1 & 1 \end{vmatrix} = 0 \implies \underline{x^2 - 2x + y^2 - 2y + z^2 = 3.}$$

(c) Graph the equation from part (c) to confirm that the sphere passes through the four given points.

## Section 4.1 Real Vector Spaces

**Objectives.**

- Introduce the vector space axioms.
- Discuss some examples of real vector spaces.

---

A vector space is a generalization of the vector arithmetic in $\mathbb{R}^n$. A (nonempty) set of objects forms a vector space if it satisfies ten assumptions (axioms) that describe the rules of arithmetic for two operations.

real number.

**Vector space axioms.** Let $V$ be a (nonempty) set of objects with two operations called *addition* and *scalar multiplication*. If the following ten axioms are satisfied by all $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $V$ and all scalars $k$ and $m$, then $V$ is a vector space.

1. If $\vec{u}$ and $\vec{v}$ are in $V$, then $\vec{u}+\vec{v}$ is in $V$. V is closed under addition.
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

   axioms 2-5 are "properties of addition"
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
4. There exists a vector $\vec{0}$ in $V$ that satisfies $\vec{0}+\vec{u}=\vec{u}+\vec{0}=\vec{u}$ for all $\vec{u}$ in $V$.
5. For each $\vec{u}$ in $V$, the vector $-\vec{u}$ (the negative of $\vec{u}$) is in $V$ and satisfies $\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=\vec{0}$.
6. If $\vec{u}$ is in $V$ and $k$ is a scalar, then $k\vec{u}$ is in $V$. V is closed under scalar multiplication.
7. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
8. $(k+m)\vec{u}=k\vec{u}+m\vec{u}$

   axioms 7-10 are "properties of scalar multiplication"
9. $k(m\vec{u})=(km)\vec{u}$
10. $1\vec{u}=\vec{u}$

**Strategy.** To show that a set $V$ with two operations is a vector space: usually $\mathbb{R}$

- identify the vectors and the scalars
- identify the operations of addition and scalar multiplication
- show axioms 1 and 6 hold (closure of V)
- show axioms 2-5 and axioms 7-10 hold.

**Example 1.** The set $V = \{\vec{0}\}$ with the operations

$$\underbrace{\vec{0}+\vec{0}=\vec{0}}_{\text{addition}} \quad \text{and} \quad \underbrace{k\vec{0}=\vec{0} \quad \text{for all scalars } k}_{\text{scalar multiplication.}}$$

is a vector space.

$V$ is closed, because $\vec{0}+\vec{0}=\vec{0}$ is in $V$ and $k\vec{0}=\vec{0}$ is in $V$.

eg. axiom 3: $\vec{0}+\underbrace{(\vec{0}+\vec{0})}_{\substack{=\vec{0} \\ \text{(by def.)}}} = \underbrace{\vec{0}}_{\substack{=\vec{0}+\vec{0} \\ \text{(by def.)}}}+\vec{0} = (\vec{0}+\vec{0})+\vec{0}$.

**Example 2.** The set $V=\mathbb{R}^n$ of all $n$-tuples of real numbers with the operations

$$\vec{u}+\vec{v}=(u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)=(u_1+v_1,u_2+v_2,\dots,u_n+v_n),$$
$$k\vec{u}=k(u_1,u_2,\dots,u_n)=(ku_1,ku_2,\dots,ku_n)$$

is a vector space.

eg. axiom 2:

$$\begin{aligned}\vec{u}+\vec{v} &= (u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)\\ &= (u_1+v_1,\, u_2+v_2,\cdots,\, u_n+v_n)\\ &= (v_1+u_1,\, v_2+u_2,\cdots,\, v_n+u_n)\\ &= (v_1,v_2,\cdots,v_n)+(u_1,u_2,\cdots,u_n)\\ &= \vec{v}+\vec{u}.\end{aligned}$$

eg. axiom 7:

$$\begin{aligned}k(\vec{u}+\vec{v}) &= k(u_1+v_1,\cdots,u_n+v_n)\\ &= (k(u_1+v_1),\cdots,k(u_n+v_n))\\ &= (ku_1+kv_1,\,\dots,\,ku_n+kv_n)\\ &= (ku_1,\cdots,ku_n)+(kv_1,\cdots,kv_n)\\ &= k(u_1,\cdots,u_n)+k(v_1,\cdots,v_n)\\ &= k\vec{u}+k\vec{v}.\end{aligned}$$

**Example 3.** The set $V=\mathbb{R}^\infty$ of all infinite sequences of real numbers with the operations

$$\vec{u}+\vec{v}=(u_1,u_2,\dots,u_n,\dots)+(v_1,v_2,\dots,v_n,\dots)=(u_1+v_1,u_2+v_2,\dots,u_n+v_n,\dots),$$
$$k\vec{u}=k(u_1,u_2,\dots,u_n,\dots)=(ku_1,ku_2,\dots,ku_n,\dots)$$

is a vector space.

eg. axiom 4: Define $\vec{0}=(0,0,\cdots,0,\cdots)$. Then $\vec{0}$ is in $V=\mathbb{R}^\infty$.
If $\vec{u}=(u_1,u_2,\cdots,u_n,\cdots)$, then

$$\vec{u}+\vec{0}=(u_1,u_2,\cdots,u_n,\cdots)+(0,0,\cdots,0,\cdots)=(u_1,u_2,\cdots,u_n,\cdots)=\vec{u}$$
$$\vec{0}+\vec{u}=(0,0,\cdots,0,\cdots)+(u_1,u_2,\cdots,u_n,\cdots)=(u_1,u_2,\cdots,u_n,\cdots)=\vec{u}.$$

**Example 4.** The set $V = M_{22}$ of all $2 \times 2$ matrices of real numbers with the operations

$$\vec{u}+\vec{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}, \leftarrow \text{closed under addition}$$

$$k\vec{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \leftarrow \text{closed under scalar multiplication}$$

is a vector space. note: the "vectors" in $M_{22}$ are 2x2 matrices.

axiom 5: Define $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (Then $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$ for all $\vec{u}$ in $M_{22}$).

For $\vec{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, define $-\vec{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$. Then:

$$\vec{u}+(-\vec{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} u_{11}-u_{11} & u_{12}-u_{12} \\ u_{21}-u_{21} & u_{22}-u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}.$$

$$(-\vec{u})+\vec{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} -u_{11}+u_{11} & -u_{12}+u_{12} \\ -u_{21}+u_{21} & -u_{22}+u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \vec{0}.$$

**Example 5.** The set $V = M_{mn}$ of all $m \times n$ matrices of real numbers

$$\vec{u} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{bmatrix}$$

with the operations of matrix addition and scalar multiplication is a vector space.

The "vectors" in $M_{mn}$ are $m \times n$ matrices of real numbers.

The "zero vector" in $M_{34}$ is

$$\vec{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

note: $F(a,b)$ is all functions defined on $(a,b)$.
$F[a,b]$ is all functions defined on $[a,b]$.



**Example 6.** Let $F(-\infty,\infty)$ be the set of all real-valued functions defined on the interval $(-\infty,\infty)$. For $\vec{f}=f(x)$ and $\vec{g}=g(x)$ in $F(-\infty,\infty)$, we define addition and scalar multiplication by

$$\vec{f}+\vec{g}=f(x)+g(x) \qquad \text{and} \qquad k\vec{f}=kf(x) \quad \text{for all scalars } k.$$

Then $V=F(-\infty,\infty)$ with these two operations is a vector space.

$f(x)+g(x)$ and $kf(x)$ are in $F(-\infty,\infty)$, so $F(-\infty,\infty)$ is closed under addition and scalar multiplication.

axiom 2: $\vec{f}+\vec{g}=f(x)+g(x)=g(x)+f(x)=\vec{g}+\vec{f}$.

axiom 4: Define $\vec{0}=0$ for all $x$ in $(-\infty,\infty)$. Then
$\vec{f}+\vec{0}=f(x)+0=f(x)=\vec{f}$, $\vec{0}+\vec{f}=0+f(x)=f(x)=\vec{f}$.

axiom 5: Define $-\vec{f}=-f(x)$. Then
$\vec{f}+(-\vec{f})=f(x)+(-f(x))=f(x)-f(x)=0=\vec{0}$.

**Example 7.** Let $V=\mathbb{R}^2$. For $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$ in $\mathbb{R}^2$, we define addition and scalar multiplication by

$$\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2) \qquad \text{and} \qquad k\vec{u}=(ku_1,0).$$

Then $V=\mathbb{R}^2$ with these two operations is **not** a vector space.

This set (with the addition and multiplication defined above) satisfies axioms 1-9, but not axiom 10.

Let $\vec{u}=(u_1,u_2)$, where $u_2\neq 0$. Then

$$1\vec{u}=\underbrace{1(u_1,u_2)=(1u_1,0)}_{\text{by def.}}=(u_1,0)\neq\vec{u}.$$

That is, there are vectors $\vec{u}$ in $\mathbb{R}^2$ where $1\vec{u}\neq\vec{u}$.

Therefore, this is not a vector space.

**Example 8.** Let $V$ be the set of all positive real numbers. For $\vec{u}=u$ and $\vec{v}=v$ in $V$, we define addition and scalar multiplication by

$$\vec{u}+\vec{v}=uv \quad \text{and} \quad k\vec{u}=u^k.$$

"scalar multiplication" in $V$ is exponentiation

Then $V$ with these two operations is a vector space.

"addition" in $V$ is multiplication of real numbers.

If $u, v$ positive then $uv$ is positive. If $u$ is positive, then $u^k$ is positive.
That is, $V$ is closed under these two operations.

axiom 4: Define $\vec{0}=1$. Then $\vec{u}+\vec{0}=u\cdot 1=u=\vec{u}$.

axiom 7: For any scalar $k$:

$$k(\vec{u}+\vec{v})=(uv)^k=(u^k)(v^k)=k\vec{u}+k\vec{v}$$

**Some properties of vector spaces.** Let $V$ be a vector space, let $\vec{u}$ be a vector in $V$, and let $k$ be a scaler. Then:

1. $0\vec{u}=\vec{0}$.
2. $k\vec{0}=\vec{0}$
3. $(-1)\vec{u}=-\vec{u}$ i.e. $-1$ times $\vec{u}$ equals the negative of $\vec{u}$.
4. If $k\vec{u}=\vec{0}$, then either $k=0$ or $\vec{u}=\vec{0}$.

**Proof of 1.**

$$\begin{aligned}
0\vec{u} &= 0\vec{u}+\vec{0} && \text{axiom 4}\\
&= 0\vec{u}+(0\vec{u}+(-0\vec{u})) && \text{axiom 5}\\
&= (0\vec{u}+0\vec{u})+(-0\vec{u}) && \text{axiom 3}\\
&= (0+0)\vec{u}+(-0\vec{u}) && \text{axiom 8}\\
&= 0\vec{u}+(-0\vec{u}) && 0+0=0\\
&= \vec{0}. && \text{axiom 5}
\end{aligned}$$

**Proof of 3.**

We need to show that $\vec{u}+(-1)\vec{u}=\vec{0}$.

i.e. $-\vec{u}$ satisfies axiom 5.

$$\begin{aligned}
\vec{u}+(-1)\vec{u} &= 1\vec{u}+(-1)\vec{u} && \text{axiom 10}\\
&= (1+(-1))\vec{u} && \text{axiom 8}\\
&= 0\vec{u} && 1+(-1)=0\\
&= \vec{0} && \text{from 1.}
\end{aligned}$$

## Section 4.2 Subspaces

**Objectives.**

- Introduce the notion of a subspace of a vector space.
- Determine whether a subset of a vector space is a subspace.
- Discuss some subspaces of real vector spaces.

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Recall that a vector space is a set $V$ that generalizes the vector arithmetic of $\mathbb{R}^n$ – vectors in $V$ can be added or scaled without leaving $V$, and these operations are consistent with the usual rules of arithmetic.

Suppose that $W$ is a set of vectors in a vector space $V$. We call $W$ a subspace of $V$ if $W$ is a vector space with the operations of addition and scalar multiplication from $V$.

i.e. a subspace is a vector space inside a larger vector space.

**Example 1.** If $V$ is any vector space, and $\vec{0}$ is the zero vector in $V$, then $W = \{\vec{0}\}$ is a subspace of $V$.

every vector in W is also in V.

why? $W \subseteq V$ and $W = \{\vec{0}\}$ is a vector space.

Six of the ten axioms for a vector space are satisfied by every subset of vectors. The four axioms that need to be checked are:

- closure under addition axiom 1
- existence of $\vec{0}$ axiom 4
- existence of negatives axiom 5
- closure under scalar multiplication. axiom 6

**Subspace Test.** If $W$ is a nonempty set of vectors in a vector space $V$, then $W$ is a subspace of $V$ if and only if both of the following conditions are satisfied.

1. If $\vec{u}$ and $\vec{v}$ are in $W$, then $\vec{u} + \vec{v}$ is in $W$.

2. If $\vec{u}$ is in $W$ and $k$ is a scalar, then $k\vec{u}$ is in $W$

**Strategy.** To show that $W$ is a subspace of $V$:

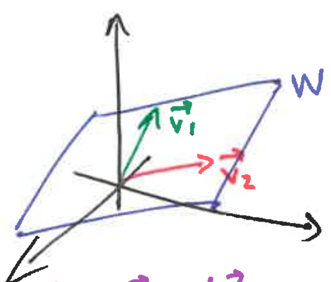
- show that if $\vec{u}_1, \vec{u}_2$ are in $W$, then $\vec{u}_1 + \vec{u}_2$ is in $W$
- show that if $\vec{u}$ is in $W$, then $k\vec{u}$ is in $W$ for all $k$.

**Example 2.** If $W$ is a line through the origin in $\mathbb{R}^n$, then $W$ is a subspace of $\mathbb{R}^n$.

Let $W$ be the line $\vec{x} = t\vec{v}$. If $\vec{u}_1 = s_1\vec{v}$ and $\vec{u}_2 = s_2\vec{v}$, then $\vec{u}_1 + \vec{u}_2 = s_1\vec{v} + s_2\vec{v} = (s_1 + s_2)\vec{v}$, so $\vec{u}_1 + \vec{u}_2$ is in $W$.

If $\vec{u} = s\vec{v}$ and $k$ is a scalar, then $k\vec{u} = k(s\vec{v}) = (ks)\vec{v}$, so $k\vec{u}$ is in $W$. Because $W$ is closed under addition and closed under scalar multiplication, $W$ is a subspace of $\mathbb{R}^n$.

**Example 3.** If $W$ is a plane through the origin in $\mathbb{R}^3$, then $W$ is a subspace of $\mathbb{R}^3$.




$\vec{x} = s\vec{v}_1 + t\vec{v}_2$

If $\vec{u}_1 = s_1\vec{v}_1 + s_2\vec{v}_2$ and $\vec{u}_2 = t_1\vec{v}_1 + t_2\vec{v}_2$, then

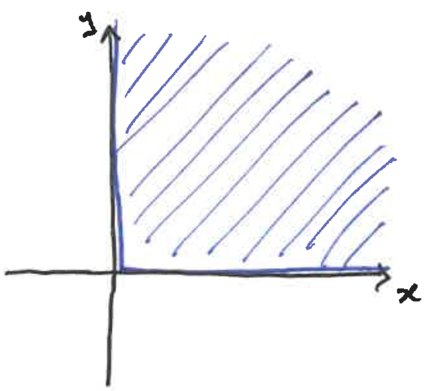
$\vec{u}_1 + \vec{u}_2 = (s_1\vec{v}_1 + s_2\vec{v}_2) + (t_1\vec{v}_1 + t_2\vec{v}_2) = (s_1 + t_1)\vec{v}_1 + (s_2 + t_2)\vec{v}_2$.

If $\vec{u} = s\vec{v}_1 + t\vec{v}_2$ and $k$ is a scalar, then

$k\vec{u} = k(s\vec{v}_1 + t\vec{v}_2) = (ks)\vec{v}_1 + (kt)\vec{v}_2$.

Thus $W$ is a subspace of $\mathbb{R}^3$.

**Example 4.** The set $W$ of all points $(x, y)$ in $\mathbb{R}^2$ with $x \geq 0$ and $y \geq 0$ is <u>not</u> a subspace of $\mathbb{R}^2$.




This set <u>is</u> closed under addition, but is <u>not</u> closed under scalar multiplication.

eg. $\vec{u} = (1, 1)$ is in $W$, but

$-1\vec{u} = (-1, -1)$ is <u>not</u> in $W$.

Thus $W$ is <u>not</u> a subspace of $\mathbb{R}^2$.

**Subspaces of $\mathbb{R}^2$.**

- $\{\vec{0}\}$
- lines through the origin
- $\mathbb{R}^2$

**Subspaces of $\mathbb{R}^3$.**

- $\{\vec{0}\}$
- lines through the origin
- planes through the origin
- $\mathbb{R}^3$

Recall that $M_{nn}$ is the vector space of all $n \times n$ matrices of real numbers.

**Example 5.** Let $W$ be the set of all symmetric $n \times n$ matrices.

(a) Discuss why $W$ is a subspace of $M_{nn}$.

The sum of two symmetric matrices is a symmetric matrix, and a scalar multiple of a symmetric matrix is symmetric.

Thus $W$ is a subspace of $M_{nn}$.

(b) What are some other subspaces of $M_{nn}$?

- diagonal matrices.
- upper triangular matrices.
- lower triangular matrices.

**Example 6.** Let $W$ be the set of all invertible $2 \times 2$ matrices.

(a) Find two matrices $A$ and $B$ in $W$ such that $A + B$ is not in $W$. (What does this example show?)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are in $W$ (because $\det \neq 0$),

but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is <u>not</u> in $W$ (b/c $\det(A+B) = 0$).

Thus $W$ is <u>not</u> closed under addition, and thus ~~W~~ is <u>not</u> a subspace of $M_{22}$.

(b) Find a matrix $A$ and a scalar $k$ such that $kA$ is not in $W$. (What does this example show?)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in $W$ (b/c $\det A \neq 0$), but $0A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is <u>not</u> in $W$ (b/c $\det(0A) = 0$).

Thus $W$ is <u>not</u> closed under scalar ~~multicaption~~ multiplication, and thus is <u>not</u> a subspace of $M_{22}$.

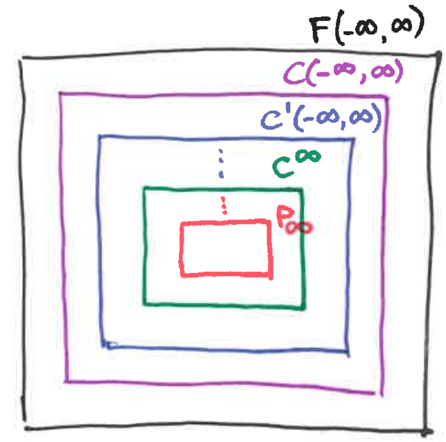
*Note: more generally, the set of all invertible $n \times n$ matrices is not a subspace of $M_{nn}$.*

Recall that $F(-\infty,\infty)$ is the set of all (real-valued) functions defined on the interval $(-\infty,\infty)$.

**Example 7.** The set $C(-\infty,\infty)$ of all *continuous* functions defined on $(-\infty,\infty)$ is a subspace of $F(-\infty,\infty)$.

If $f(x)$ and $g(x)$ are continuous, then $f(x)+g(x)$ and $kf(x)$ are also continuous.

**Example 8.** The following sets of functions are subspaces of $F(-\infty,\infty)$.




(a) $C^1(-\infty,\infty)$ • all f^n^s f where the derivative is continuous.

(b) $C^m(-\infty,\infty)$ • all f^n^s f where the first m derivatives are continuous.

m is a positive integer

(c) $C^\infty(-\infty,\infty)$ • all f^n^s f where every derivative is continuous.

A <u>polynomial of degree $n$</u> is a function that can be written

$$p(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$.

**Example 9.** The set $P_\infty$ of all polynomials is a subspace of $F(-\infty,\infty)$.

If $p(x)$ and $q(x)$ are polynomials, then $p(x)+q(x)$ and $kp(x)$ are both polynomials. Thus $P_\infty$ is a subspace of $F(-\infty,\infty)$.

**Example 10.** The set $P_n$ of all polynomials with degree at most $n$ is a subspace of $F(-\infty,\infty)$.

If $p(x), q(x)$ are polynomials with degree $\leq n$, then $p(x)+q(x)$ and $kp(x)$ are polynomials with degree $\leq n$.

<u>note</u>: If $p(x) = 1-2x^2$, $q(x) = 1+2x^2$, then $p(x)+q(x) = 2$ has degree $< 2$.

Thus the set of polynomials with degree $n$ is <u>not</u> a subspace of $F(-\infty,\infty)$.

**Example 11.** Determine whether each set of matrices is a subspace of $M_{22}$.

(a) The set $U$ of all matrices of the form $\begin{bmatrix} x & 2x \\ 0 & y \end{bmatrix}$. Let $A = \begin{bmatrix} a & 2a \\ 0 & b \end{bmatrix}$, $B = \begin{bmatrix} c & 2c \\ 0 & d \end{bmatrix}$. Then:

$A + B = \begin{bmatrix} a+c & 2(a+c) \\ 0 & b+d \end{bmatrix}$ is in $U$ (take $x = a+c$, $y = b+d$), and

$kA = \begin{bmatrix} ka & 2ka \\ 0 & kb \end{bmatrix}$ is in $U$ (take $x = ka$, $y = kb$).

Thus $U$ is a subspace of $M_{22}$.

(b) The set $W$ of all $2 \times 2$ matrices $A$ such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Then: $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $A$ is in $W$.

But $(2A)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, so $2A$ is <u>not</u> in $W$.

Thus $W$ is <u>not</u> closed under scalar multiplication, so is <u>not</u> a subspace of $M_{22}$.

**Example 12.** Determine whether each set of polynomials is a subspace of $P_2$.

(a) The set $U$ of all polynomials of the form $p(x) = 1 - ax + ax^2$.

If $p(x) = 1 - x + x^2$ and $q(x) = 1 - 2x + 2x^2$, then $p, q$ are in $U$

but $p(x) + q(x) = 2 - 3x + 3x^2$ is <u>not</u> in $U$.

Thus $U$ is <u>not</u> a subspace of $P_2$.

(b) The set $W$ of all polynomials such that $p(3) = 0$.

If $p, q$ satisfy $p(3) = 0$ and $q(3) = 0$, then

$(p+q)(3) = p(3) + q(3) = 0 + 0 = 0$, and

$(kp)(3) = k\,p(3) = k \cdot 0 = 0$.

Thus $p+q$ and $kp$ are in $W$, so $W$ is a subspace of $P_2$.

**Theorem.** If $W_1, W_2, \ldots, W_k$ are all subspaces of a vector space $V$, then the set $W$ of all vectors in the intersection of these subspaces is a subspace of $V$.

all vectors in every subspaces $W_1, W_2, \ldots, W_k$.

**Theorem.** Let $A$ be an $m \times n$ matrix. The set of all solutions $\vec{x}$ to the homogeneous linear system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

why? If $A\vec{x}_1 = \vec{0}$ and $A\vec{x}_2 = \vec{0}$, then $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}$
and $A(k\vec{x}_1) = k(A\vec{x}_1) = k\vec{0} = \vec{0}$.

The solution space in the previous theorem is called the kernel of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

**Theorem.** Let $A$ be an $m \times n$ matrix. Then the kernel of the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$.

**Example 13.** Describe the geometry of the solution space for each homogeneous linear system.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = 2s - 3t$, $y = s$, $z = t$.
This is a plane through the origin in $\mathbb{R}^3$ with normal vector $(1, -2, 3)$.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = -5t$, $y = -t$, $z = t$.
This is a line through the origin in $\mathbb{R}^3$ parallel to $\vec{v} = (-5, -1, 1)$.

(c) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is $x = 0$, $y = 0$, $z = 0$.
This is the point at the origin.

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution space is all $(x, y, z)$ in $\mathbb{R}^3$.

## Section 4.3 Spanning Sets

**Objectives.**

- Introduce the span of a set of vectors.
- Define spanning sets for a subspace of a vector space.
- Discuss examples of spanning sets in real vector spaces.

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Let $V$ be a vector space, and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ be vectors in $V$. The vector $\vec{w}$ in $V$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ if there are scalars $k_1, k_2, \ldots, k_r$ such that

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r.$$

**Theorem.** If $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r\}$ is a nonempty set of vectors in a vector space $V$, then:

(a) The set $W$ of all linear combinations of vectors in $S$ is a subspace of $V$. $W = \text{Span}(S)$.
(also, $W$ is "spanned" by $S$).

(b) The set $W$ in part (a) is the smallest subspace of $V$ that contains all the vectors in $S$.
*(This means that any other subspace of $V$ that contains $S$ also contains every vector in $W$.)*

**Proof of (a).** Let $\vec{u} = a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_r\vec{w}_r$, $\vec{v} = b_1\vec{w}_1 + b_2\vec{w}_2 + \cdots + b_r\vec{w}_r$.

Then: $\vec{u} + \vec{v} = (a_1+b_1)\vec{w}_1 + (a_2+b_2)\vec{w}_2 + \cdots + (a_r+b_r)\vec{w}_r$,

$k\vec{u} = (ka_1)\vec{w}_1 + (ka_2)\vec{w}_2 + \cdots + (ka_r)\vec{w}_r$.

Because $W$ is closed under addition and scalar multiplication, $W$ is a subspace of $V$.

**Proof of (b).** If $W'$ is a subspace of $V$ that contains $S$, then $W'$ is closed under addition and scalar multiplication. Thus $W'$ contains all linear combinations of vectors in $S$, so $W'$ contains $W$.

The subspace $W$ in this theorem is the called subspace of $V$ spanned by $S$, and we say that the vectors in $S$ span the subspace $W$.

**Example 1.** Every vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$.

$\vec{e}_1 = (1,0,0,\cdots,0)$

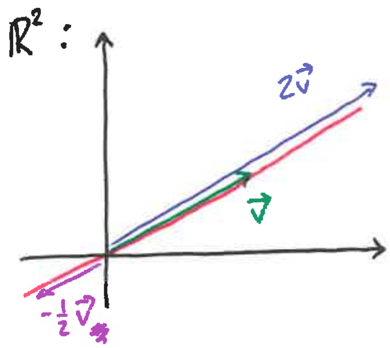
$\vec{e}_2 = (0,1,0,\cdots,0)$

Let $\vec{v} = (v_1, v_2, \cdots, v_n)$ be a vector in $\mathbb{R}^n$.

Then $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$.

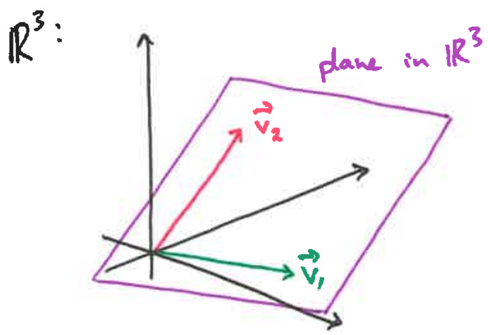
That is, $\vec{v}$ is in $\text{span}\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$.

**Example 2.** (a) Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. Describe $\text{span}\{\vec{v}\}$.




$\text{span}\{\vec{v}\}$ is the set of all scalar multiples of $\vec{v}$. Thus $\text{span}\{\vec{v}\}$ is the line through the origin parallel to $\vec{v}$.

(b) Let $\vec{v}_1$ and $\vec{v}_2$ be non-parallel vectors in $\mathbb{R}^3$. Describe $\text{span}\{\vec{v}_1, \vec{v}_2\}$.




Every vector $k_1\vec{v}_1 + k_2\vec{v}_2$ lies in the plane determined by $\vec{v}_1$ and $\vec{v}_2$.

Thus $\text{span}\{\vec{v}_1, \vec{v}_2\}$ is the plane through the ~~par~~ origin and parallel to both $\vec{v}_1$ and $\vec{v}_2$.

**Example 3.** Every polynomial in $P_n$ can be written as a linear combination of the polynomials $1, x, x^2, \ldots, x^n$.

all polynomials of degree $\leq n$.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ ← arbitrary polynomial in $P_n$

$= a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n)$.

Thus $p(x)$ is in $\text{span}\{1, x, x^2, \cdots, x^n\}$.

Therefore $P_n = \text{span}\{1, x, x^2, \ldots, x^n\}$.

There are two important problems we can ask about spanning sets in a vector space.

- Given a set of vectors $S$ and a vector $\vec{v}$, decide whether $\vec{v}$ is in $\text{span}(S)$.
- Given a set of vectors $S$, decide whether $\text{span}(S) = V$.

← can $\vec{v}$ be written as a linear combination of vectors in $S$?

**Example 4.** Let $\vec{u} = (1, 2, -1)$ and $\vec{v} = (6, 4, 2)$.

(a) Show that $\vec{w_1} = (9, 2, 7)$ is a linear combination of $\vec{u}$ and $\vec{v}$.

The equation $(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$ is equivalent to the linear system: $9 = k_1 + 6k_2$, $2 = 2k_1 + 4k_2$, $7 = -k_1 + 2k_2$.

This system has solution $k_1 = -3$, $k_2 = 2$.

Thus $\vec{w}_1 = -3\vec{u} + 2\vec{v}$. (i.e. $\vec{w}_1$ is in $\text{span}\{\vec{u}, \vec{v}\}$.)

(b) Show that $\vec{w_2} = (4, -1, 8)$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

The equation $(4,-1,8) = k_1(1,2,-1) + k_2(6,4,2)$ is equivalent to the linear system: $4 = k_1 + 6k_2$, $-1 = 2k_1 + 4k_2$, $8 = -k_1 + 2k_2$.

This system is inconsistent (i.e. no solutions!!!), so $\vec{w}_2$ is <u>not</u> a linear combination of $\vec{u}$ and $\vec{v}$. (i.e $\vec{w}_2$ is <u>not</u> in $\text{span}\{\vec{u}, \vec{v}\}$.)

**Example 5.** Determine whether the vectors $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (1, 0, 1)$, and $\vec{v}_3 = (2, 1, 3)$ span $\mathbb{R}^3$.

We need to decide whether <u>every</u> vector $(b_1, b_2, b_3)$ is in $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$(b_1, b_2, b_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3) = (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$$

This is equivalent to the linear system:

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \phantom{+k_2} + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned} \implies \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Because $\det \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} = 0$ (check this!!!), this system is inconsistent for some choices of $b_1, b_2, b_3$. Thus $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is <u>not</u> $\mathbb{R}^3$.

**Strategy.** To determine whether the set $S = \{\vec{w}_1, \ldots, \vec{w}_r\}$ spans the vector space $V$:

- choose an arbitrary vector $\vec{v}$ in $V$.
- set up a linear system from $\vec{v} = k_1\vec{w}_1 + \cdots + k_r\vec{w}_r$.
- decide whether the linear system is consistent for all $\vec{v}$ in $V$.

**Example 6.** Determine whether the set $S$ spans $P_2$.

(a) $S = \{1 + x + x^2, -1 - x, 2 + 2x + x^2\}$

Let $p(x) = a + bx + cx^2$. The equation

$a + bx + cx^2 = k_1(1 + x + x^2) + k_2(-1-x) + k_3(2 + 2x + x^2)$ is equivalent to

$$\begin{cases} k_1 - k_2 + 2k_3 = a \\ k_1 - k_2 + 2k_3 = b \\ k_1 \qquad\quad + k_3 = c. \end{cases}$$

Because $\begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0$, the system is inconsistent for some choices of $a, b, c$.

Thus $S$ does not span $P_2$.

(b) $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$

Let $p(x) = a + \cancel{bx} + cx^2$. The equation

$a + bx + cx^2 = k_1(x + x^2) + k_2(x - x^2) + k_3(1 + x) + k_4(1 - x)$

is equivalent to the linear system

$$\begin{cases} \qquad\qquad k_3 + k_4 = a \\ k_1 + k_2 + k_3 - k_4 = b \\ k_1 - k_2 \qquad\qquad = c \end{cases} \Rightarrow \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

The rref for this system is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & 0 & \frac{-a+b-c}{2} \\ 0 & 0 & 1 & -1 & a \end{array}\right]$.

This is consistent for all choices of $a, b, c$, so $S$ spans $P_2$.

**Example 7.** Determine whether the set $S$ spans $M_{22}$.

(a) $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k_1 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a = k_1 + k_2 + k_3 + k_4 \\ b = 2k_1 + 2k_3 + k_4 \\ c = k_3 + k_4 \\ d = k_1 + k_2 + k_4 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Because $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = -2 \neq 0$, this system is consistent for <u>all</u> choices of $a,b,c,d$. Thus $\text{span}\{S\} = M_{22}$.

(b) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} a = k_1 - k_2 \\ b = k_4 \\ c = k_2 + k_3 - k_4 \\ d = k_4 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

~~The~~ Because $\begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0$ $(R_2 = R_4)$, this system is inconsistent for <u>some</u> choices of $a,b,c,d$.

Thus $\text{span}\{S\} \neq M_{22}$.

## Section 4.4 Linear Independence

**Objectives.**

- Define linear independence of vectors.
- Determine whether a set of vectors is linearly independent or linearly dependent.
- Define and apply the Wronskian to determine whether a set of functions is linearly independent.

---

Let $V$ be a vector space. A nonempty set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in $V$ is linearly independent if no vector in $S$ can be written as a linear combination of the other vectors in $S$. Otherwise, $S$ is linearly dependent.

*Note: If $S = \{\vec{v}\}$ contains one vector, then $S$ is linearly independent if $\vec{v} \neq \vec{0}$ and linearly dependent if $\vec{v} = \vec{0}$.*

**Theorem.** A nonempty set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in $V$ is linearly independent if and only if the only solution to the equation

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0}$$

is $k_1 = k_2 = \cdots = k_r = 0$.

**Example 1.** The set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of standard unit vectors in $\mathbb{R}^n$ is linearly independent.

Because $k_1\vec{e}_1 + k_2\vec{e}_2 + \cdots + k_n\vec{e}_n = (k_1, k_2, \cdots, k_n)$, the only solution to $k_1\vec{e}_1 + \cdots + k_n\vec{e}_n = \vec{0}$ is $k_1 = 0, k_2 = 0, \cdots, k_n = 0$.

Thus $\{\vec{e}_1, \cdots, \vec{e}_n\}$ is linearly independent in $\mathbb{R}^n$.

**Example 2.** Determine whether the vectors $\vec{v}_1 = (1, -2, 3)$, $\vec{v}_2 = (5, 6, -1)$, $\vec{v}_3 = (3, 2, 1)$ are linearly independent in $\mathbb{R}^3$.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0} \Rightarrow k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0)$$

$$\Rightarrow \begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases} \Rightarrow k_1 = -\tfrac{1}{2}t,\ k_2 = -\tfrac{1}{2}t,\ k_3 = t.$$

or: $\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 0.$

This system has non-trivial solutions, so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

**Example 3.** Determine whether the vectors $\vec{v}_1 = (1,2,2,-1)$, $\vec{v}_2 = (4,9,9,-4)$, $\vec{v}_3 = (5,8,9,-5)$ are linearly independent in $\mathbb{R}^4$.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0} \Rightarrow k_1(1,2,2,-1) + k_2(4,9,9,-4) + k_3(5,8,9,-5) = \vec{0}$$

$$\Rightarrow \begin{cases} k_1 + 4k_2 + 5k_3 = 0 \\ 2k_1 + 9k_2 + 8k_3 = 0 \\ 2k_1 + 9k_2 + 9k_3 = 0 \\ -k_1 - 4k_2 - 5k_3 = 0 \end{cases} \Rightarrow k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

Gaussian elimination.

(note: cannot use determinant, because the coefficient matrix is not square)

Thus $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

**Example 4.** The set $\{1, x, x^2, \ldots, x^n\}$ of polynomials in $P_n$ is linearly independent.

If $a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n) = 0$, then $a_0 = a_1 = \cdots = a_n = 0$.

Thus $\{1, x, x^2, \cdots, x^n\}$ is linearly independent in $P_n$.

**Example 5.** Determine whether the polynomials $p_1(x) = 1 - x$, $p_2(x) = 5 + 3x - 2x^2$, $p_3(x) = 1 + 3x - x^2$ are linearly independent in $P_2$.

$$k_1p_1(x) + k_2p_2(x) + k_3p_3(x) = 0 \Rightarrow k_1(1-x) + k_2(5+3x-2x^2) + k_3(1+3x-x^2) = 0.$$

$$\Rightarrow \begin{cases} k_1 + 5k_2 + k_3 = 0 & \text{(from constant terms)} \\ -k_1 + 3k_2 + 3k_3 = 0 & \text{(from linear terms)} \\ -2k_2 - k_3 = 0 & \text{(from quadratic terms)} \end{cases}$$

Because $\begin{vmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{vmatrix} = 0$, this system has non-trivial solutions.

Therefore, $p_1(x), p_2(x), p_3(x)$ are linearly dependent.

**Theorem.** Let $S$ be a nonempty set of vectors in a vector space $V$.

(a) If $\vec{0}$ is in $S$ then $S$ is linearly dependent.

(b) If $S$ contains exactly two vectors, then $S$ is linearly independent if and only if neither vector is a scalar multiple of the other.

i.e. $\vec{u} = k\vec{v} \iff \vec{u}, \vec{v}$ are linearly dependent.

**Example 6.** Recall that $F(-\infty, \infty)$ is the set of all functions defined on $(-\infty, \infty)$.

(a) Show that the functions $f(x) = x$ and $g(x) = \cos x$ are linearly independent in $F(-\infty, \infty)$

$f(x)$ is not a scalar multiple of $g(x)$, so $f$ and $g$ are linearly independent in $F(-\infty,\infty)$.

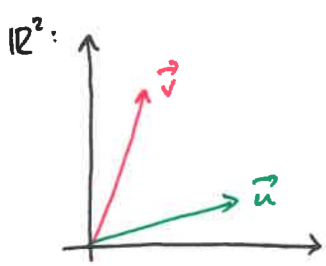
(b) Show that the functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent in $F(-\infty, \infty)$

$f(x) = \sin 2x = 2 \sin x \cos x = 2\,g(x)$

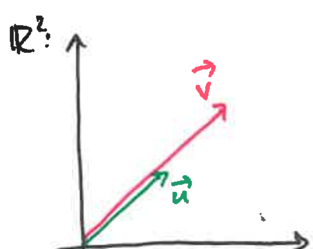
Because $f$ is a scalar multiple of $g$, the functions $f$ and $g$ are linearly dependent in $F(-\infty,\infty)$.

The second condition in the previous theorem can be interpreted – and extended – geometrically as follows.

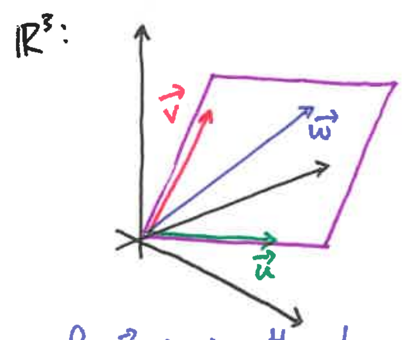
- Two distinct nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are linearly dependent if and only if they are parallel – that is, they lie on the same line.
- Three distinct nonzero vectors in $\mathbb{R}^3$ are linearly dependent if and only if they lie in the same plane.




linearly independent




linearly dependent




if $\vec{w}$ is in the plane with $\vec{u},\vec{v}$ then $\vec{u},\vec{v},\vec{w}$ are linearly dependent.

**Theorem.** Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ be a nonempty set of vectors ~~vectors~~ in $\mathbb{R}^n$. If $r > n$ then $S$ is linearly dependent.

This says that a linearly independent set in $\mathbb{R}^n$ contains at most $n$ vectors.

eg. $\{(0,1), (2,-1), (1,3)\}$ is linearly dependent in $\mathbb{R}^2$.

note: $7(0,1) + 1(2,-1) - 2(1,3) = (0,0)$.

Our first methods of solving a linear system involved reduction of the coefficient matrix to (reduced) row echelon form. The next example demonstrates a general principle about matrices in ref and rref:

If an (augmented) matrix is in ref (rref) then the set of nonzero rows is linearly independent.

**Example 7.** Let $A = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 1 & a_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and let $\vec{r}_1 = (1, a_{12}, a_{13}, a_{14})$, $\vec{r}_2 = (0,0,1,a_{24})$, $\vec{r}_3 = (0,0,0,1)$.

Show that the equation $c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = \vec{0}$ has only the trivial solution $c_1 = c_2 = c_3 = 0$.

$$c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = \vec{0} \Rightarrow c_1(1, a_{12}, a_{13}, a_{14}) + c_2(0,0,1,a_{24}) + c_3(0,0,0,1) = \vec{0}$$

$$\Rightarrow \begin{cases} c_1 = 0 \\ c_1 a_{12} = 0 \\ c_1 a_{13} + c_2 = 0 \\ c_1 a_{14} + c_2 a_{24} + c_3 = 0 \end{cases} \Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0$$

Thus $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are linearly independent.

Given functions $f_1(x), f_2(x), \ldots, f_n(x)$ that are differentiable $n-1$ times on $(-\infty, \infty)$, the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

"differentiate n-1 times"

is the Wronskian of $f_1, f_2, \ldots, f_n$.

**Theorem.** If the Wronskian of the functions $f_1, f_2, \ldots, f_n$ is not identically zero on $(-\infty, \infty)$, then the functions are linearly independent.

note: The converse is not true!!!
That is, if $W(x) = 0$ for all $x$, then $f_1, \ldots, f_n$ could be either linearly independent or linearly dependent.

**Example 8.** Show that $f(x) = x$ and $g(x) = \cos x$ are linearly independent in $C^\infty(-\infty, \infty)$.

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x\sin x - \cos x.$$

Because $W(x)$ is not identically zero (eg. $W(0) = -1$), $f(x)$ and $g(x)$ are linearly independent.

**Example 9.** Show that $f_1(x) = 1$, $f_2(x) = e^x$, $f_3(x) = e^{2x}$ are linearly independent in $C^\infty(-\infty, \infty)$.

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} = 4e^{3x} - 2e^{3x} = 2e^{3x}.$$

Because $W(x)$ is not identically zero (eg. $W(0) = 2$), the functions $f_1, f_2, f_3$ are linearly independent in $C^\infty(-\infty, \infty)$.

## Section 4.5 Coordinates and Basis

**Objectives.**

- Introduce the idea of a basis for a vector space.
- Find coordinates for a vector relative to a given basis.

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A vector space $V$ is finite-dimensional if there is a finite set of vectors $S$ that spans $V$. Otherwise, $V$ is infinite-dimensional.

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in a finite-dimensional vector space $V$. We say that $S$ is a basis for $V$ if the following two conditions hold.

- $S$ spans $V$ i.e. every vector in $V$ is a linear combination of vectors in $S$.
- $S$ is linearly independent i.e. if $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n = \vec{0}$ then $k_1 = k_2 = \cdots = k_n = 0$.

**Example 1.** The set $\boxed{S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}}$ is a basis for $\mathbb{R}^n$. "standard basis for $\mathbb{R}^n$"

From Example 1, Section 4.3, $\mathbb{R}^n = \text{span}(S)$.

From Example 1, Section 4.4, $S$ is linearly independent.

Therefore, $S$ is a basis for $\mathbb{R}^n$.

**Example 2.** The set $\boxed{S = \{1, x, x^2, \ldots, x^n\}}$ is a basis for $P_n$. "standard basis for $P_n$"

From Example 3, Section 4.3, $P_n = \text{span}(S)$.

From Example 4, Section 4.4, $S$ is linearly independent.

Therefore, $S$ is a basis for $P_n$.

**Example 3.** The vector space $P_\infty$ is infinite-dimensional.

If $S = \{p_1, p_2, \cdots, p_r\}$ is a finite set of polynomials, then ~~there~~ $S$ contains a polynomial of maximum degree, say degree $n$. Then any linear combination of polynomials in $S$ has degree at most $n$. Thus we cannot express $x^{n+1}$ as a linear combination of polynomials in $S$, so $S$ does not span $P_\infty$.

Therefore, $P_\infty$ is infinite-dimensional.



note: $F(-\infty, \infty)$ is also infinite-dimensional.

**Example 4.** Show that the vectors $\vec{v}_1 = (1,2,1)$, $\vec{v}_2 = (2,9,0)$, $\vec{v}_3 = (3,3,4)$ form a basis for $\mathbb{R}^3$.

- linear independence:

If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, then

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 \qquad + 4c_3 &= 0. \end{aligned}$$

- spanning set:

If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = (b_1, b_2, b_3)$, then

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad + 4c_3 &= b_3. \end{aligned}$$

Because $\det\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} = -1 \neq 0$, the homogeneous system has only the trivial solution (so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent), and the nonhomogeneous is consistent for all vectors $(b_1, b_2, b_3)$ in $\mathbb{R}^3$.

Therefore, $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

**Example 5.** Show that the matrices $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis for the vector space $M_{22}$.

- linear independence:

If $c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, so $M_1, M_2, M_3, M_4$ are linearly independent.

- spanning set:

If $c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Taking $c_1 = a$, $c_2 = b$, $c_3 = c$, $c_4 = d$ satisfies this equation, so $M_1, M_2, M_3, M_4$ span $M_{22}$.

Therefore, $\boxed{S = \{M_1, M_2, M_3, M_4\}}$ is a basis for $M_{22}$.

"standard basis for $M_{22}$"

**Theorem.** Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for the vector space $V$. Then every vector $\vec{v}$ in $V$ can be written as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

in exactly one way.

**Proof.** Because $S$ is a basis for $V$, every vector in $V$ can be written as a linear combination of vectors in $S$.

Suppose $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ and $\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$.

Then $\vec{0} = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \cdots + (c_n - d_n)\vec{v}_n$.

Because $S$ is linearly independent, we have $c_1 - d_1 = 0$, $c_2 - d_2 = 0$, $\cdots$, $c_n - d_n = 0$.

Thus $c_1 = d_1$, $c_2 = d_2$, $\cdots$, $c_n = d_n$.

Therefore, $\vec{v}$ ~~be~~ can be written as a linear combination of the basis $S$ in exactly one way.

The numbers $c_1, c_2, \ldots, c_n$ in this theorem are called the coordinates of $\vec{v}$ relative to the basis $S$. The vector $(c_1, c_2, \ldots, c_n)$ is called the coordinate vector of $\vec{v}$ relative to the basis $S$, and is denoted by

$$(\vec{v})_S = (c_1, c_2, \ldots, c_n).$$

If $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$, then

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \iff (\vec{v})_S = (c_1, c_2, \cdots, c_n).$$

**Example 6.** Consider the standard basis $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for $\mathbb{R}^3$. What is the coordinate vector for $\vec{v} = (a, b, c)$ relative to the basis $S$?

$\vec{v} = (a, b, c) = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$, so $(\vec{v})_S = (a, b, c)$.

**Example 7.** Consider the basis $S = \{(1, 0), (1, 2)\}$ for $\mathbb{R}^2$. What is the coordinate vector for $\vec{v} = (-1, 4)$ relative to the basis $S$?

$\vec{v} = (-1, 4) = -3(1, 0) + 2(1, 2)$, so $(\vec{v})_S = (-3, 2)$.

**Example 8.** Find the coordinate vector for the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ relative to the standard basis for $P_n$.

$$p(x) = a_0(1) + a_1(x) + a_2(x^2) + \cdots + a_n(x^n),$$

$$\text{so} \quad (p(x))_S = (a_0, a_1, a_2, \ldots, a_n).$$

**Example 9.** Find the coordinate vector for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ relative to the standard basis for $M_{22}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\text{so} \quad (A)_S = (a, b, c, d).$$

**Example 10.** Recall from Example 4 that $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (2, 9, 0)$, $\vec{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

(a) Find the coordinate vector for $\vec{v} = (5, -1, 9)$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

From $(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$, we obtain

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 \qquad\quad + 4c_3 = 9 \end{cases}.$$

The solution is $c_1 = 1$, $c_2 = -1$, $c_3 = 2$.

Therefore, $(\vec{v})_S = (1, -1, 2)$.

(b) Find $\vec{w}$ given that $(\vec{w})_S = (-1, 3, 2)$.

$$\vec{w} = -1\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_3 = -1(1,2,1) + 3(2,9,0) + 2(3,3,4)$$

$$= \underline{(11, 31, 7)}.$$

## Section 4.6 Dimension

**Objectives.**

- Define the dimension of a finite-dimensional vector space.
- Relate dimension to span and linear independence.

**Theorem.** Every basis for a finite-dimensional vector space $V$ contains the same number of vectors.

The number of vectors in a basis for the finite-dimensional vector space $V$ is called the dimension of $V$, and is denoted by $\dim V$.

note: if $V = \{\vec{0}\}$, then $\dim V = 0$.

**Example 1.** What is the dimension of each vector space?

(a) $\mathbb{R}^n$ The standard basis is $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$.

Thus $\dim(\mathbb{R}^n) = n$.

(b) $P_n$ The standard basis is $\{1, x, \dots, x^n\}$.

Thus $\dim(P_n) = n+1$.

(c) $M_{mn}$

$\dim(M_{mn}) = mn$. e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in the standard basis for $M_{23}$.

**Theorem.** Let $V$ be a finite-dimensional vector space with $\dim V = n$.

1. If $W$ is a subset of $V$ that contains more than $n$ vectors, then $W$ is linearly dependent.
2. If $W$ is a subset of $V$ that contains fewer than $n$ vectors, then $W$ does not span $V$.

**Example 2.** Suppose that $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is a linearly independent set of vectors in a vector space $V$. What is $\dim(\operatorname{span}(S))$? Why?

$S$ is linearly independent, and $S$ spans span$(S)$. This means that $S$ is a basis for span$(S)$, so $\dim(\operatorname{span}(S)) = r$.

**Example 3.** Consider the linear system below. (This is Example 5 from the Section 1.2 lecture notes.)

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \quad\quad + 2x_5 \quad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \quad\quad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \quad\quad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

The general solution to this system is

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0.$$

(a) Write the solution in vector form.

$$\vec{x} = (-3r-4s-2t, r, -2s, s, t, 0)$$

$$= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0).$$

these vectors are linearly independent!!!

(b) Find a basis for the solution space of the system.

Every vector $\vec{x}$ in the solution space is a linear combination of $(-3,1,0,0,0,0)$, $(-4,0,-2,1,0,0)$, $(-2,0,0,0,1,0)$, and these vectors are linearly independent. Thus $\{(-3,1,0,0,0,0), (-4,0,-2,1,0,0), (-2,0,0,0,1,0)\}$ is a basis for the solution space.

(c) What is the dimension of the solution space?

There are three vectors in <u>any</u> basis, so this space has dimension 3.

"union" (i.e. add $\vec{v}$ to the set $S$).

**Theorem.** Let $S$ be a set of vectors in a vector space $V$.

1. If $S$ is linearly independent, and $\vec{v}$ is not in $\operatorname{span}(S)$, then $S \cup \{\vec{v}\}$ is linearly independent.
   i.e. adding a vector outside span(S) does not affect linear independence.
2. If $\vec{v}$ is in $S$, and $\vec{v}$ can be written as a (nonzero) linear combination of other vectors in $S$, then

   remove $\vec{v}$ from $S$.

$$\operatorname{span}(S) = \operatorname{span}(S - \{\vec{v}\}).$$

i.e. removing linearly dependent vectors does not affect span(S).

**Example 4.** Explain why the polynomials $p(x) = 1 + x^2$, $q(x) = 2 + x^2$, $r(x) = x^3$ are linearly independent.

$p(x)$ and $q(x)$ are linearly independent (neither is a multiple of the other). Also, $r(x)$ is not in span$\{p(x), q(x)\}$, because $r$ is cubic but $p, q$ are quadratic. Thus $\{p(x), q(x), r(x)\}$ is linearly independent.

**Theorem.** Let $V$ be a vector space with $\dim V = n$, and let $S$ be a set of $n$ vectors in $V$.

equal!!!

1. $S$ is a basis for $V$ if and only if $S$ is linearly independent.
2. $S$ is a basis for $V$ if and only if $S$ spans $V$.

**Example 5.** Explain why each set of vectors is a basis for the given vector space.

(a) $\vec{v}_1 = (1, 4)$ and $\vec{v}_2 = (3, -2)$ in $\mathbb{R}^2$

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent, and $\dim(\mathbb{R}^2) = 2$.

Thus $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$.

(b) $\vec{v}_1 = (1, 0, 2)$, $\vec{v}_2 = (-1, 0, 1)$, and $\vec{v}_3 = (2, -2, 3)$ in $\mathbb{R}^3$

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent in the $xz$-plane.

because the $y$-coord. is $\neq 0$.

Because $\vec{v}_3$ is not in the $xz$-plane (i.e. $\vec{v}_3$ is not in span$\{\vec{v}_1, \vec{v}_2\}$), the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Also, $\dim(\mathbb{R}^3) = 3$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

**Theorem.** Let $V$ be a vector space with $\dim V = n$, and let $S$ be a set of vectors in $V$.

1. If $S$ spans $V$ but is not a basis for $V$, then $S$ can be reduced to a basis for $V$ by removing some vectors.
2. If $S$ is linearly independent but is not a basis for $V$, then $S$ can be enlarged to a basis for $V$ by adding some vectors.

**Example 6.** (a) Find a subset of $S = \{(1,-1), (-1,1), (1,1)\}$ that is a basis for $\mathbb{R}^2$.

$\dim(\mathbb{R}^2) = 2$, so we need two vectors from $S$ to form a basis.

The vectors $(1,-1)$ and $(1,1)$ are linearly independent.

Thus $\{(1,-1),(1,1)\}$ is a basis for $\mathbb{R}^2$.

note: $\{(-1,1),(1,1)\}$ is also a basis for $\mathbb{R}^2$, but

$\{(1,-1),(-1,1)\}$ is not a basis for $\mathbb{R}^2$. (why?)

(b) Enlarge the set $S = \{(1,1,0), (1,0,-1)\}$ to a basis for $\mathbb{R}^3$.

Let's try adding $(1,0,0)$ to $S$.

$k_1(1,1,0) + k_2(1,0,-1) + k_3(1,0,0) = (0,0,0)$

$\Rightarrow (k_1+k_2+k_3,\ k_1,\ -k_2) = (0,0,0)$

$\Rightarrow k_1 = k_2 = k_3 = 0$

Thus $\{(1,1,0),(1,0,-1),(1,0,0)\}$ is linearly independent and contains three vectors, so this is a basis for $\mathbb{R}^3$.

note: $\{(1,1,0),(1,0,-1),(0,1,0)\}$ and $\{(1,1,0),(1,0,-1),(0,0,1)\}$ are also bases for $\mathbb{R}^3$.

**Theorem.** If $W$ is a subspace of a finite-dimensional vector space $V$, then:

1. $W$ is finite-dimensional.
2. $\dim W \leq \dim V$.
3. $W = V$ if and only if $\dim W = \dim V$.

## Section 4.7 Change of Basis

**Objectives.**

- Introduce the 'change of basis problem'.
- Define the transition matrix for a change of basis.
- Find the transition matrix for a change of basis.

---

Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for a vector space $V$, and let $\vec{v}$ be a vector in $V$. Recall the definition of the coordinate vector for $\vec{v}$ relative to $B$:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \iff [\vec{v}]_B = (c_1, c_2, \cdots, c_n) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

The set of all such coordinate vectors for $V$ is a function from $V$ to $\mathbb{R}^n$ called the coordinate map relative to $B$.

i.e. for each vector $\vec{v}$ in $V$ and each basis $B$ for $V$, there is a coordinate vector $[\vec{v}]_B$ in $\mathbb{R}^n$.

Sometimes we may want to change from one basis $B$ for $V$ to a different basis $B'$. Thus we would like to know how $[\vec{v}]_B$ and $[\vec{v}]_{B'}$ are related.

Suppose that $B = \{\vec{u}_1, \vec{u}_2\}$ and $B' = \{\vec{u}_1', \vec{u}_2'\}$ are both bases for $V$, and that $\vec{v}$ is a vector in $V$.

Let $[\vec{u}_1]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix}$, $[\vec{u}_2]_{B'} = \begin{bmatrix} c \\ d \end{bmatrix}$, and $[\vec{v}]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

Then: $\vec{u}_1 = a\vec{u}_1' + b\vec{u}_2'$ and $\vec{u}_2 = c\vec{u}_1' + d\vec{u}_2'$, so

$$\vec{v} = k_1\vec{u}_1 + k_2\vec{u}_2 = k_1(a\vec{u}_1' + b\vec{u}_2') + k_2(c\vec{u}_1' + d\vec{u}_2') = (k_1a + k_2c)\vec{u}_1' + (k_1b + k_2d)\vec{u}_2'$$

Thus: $$[\vec{v}]_{B'} = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{\text{transition matrix from } B \text{ to } B'}[\vec{v}]_B.$$

$$P_{B\to B'} = \big[\,[\vec{u}_1]_{B'} \;\; [\vec{u}_2]_{B'}\,\big].$$

> **Change of Basis Problem.** Suppose that $B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is the old basis for $V$, and $B' = \{\vec{u}_1', \vec{u}_2', \ldots, \vec{u}_n'\}$ is the new basis for $V$. Then the coordinate vectors for a vector $\vec{v}$ in $V$ satisfy
>
> $$[\vec{v}]_{B'} = P_{B\to B'}\,[\vec{v}]_B$$
>
> where $P_{B\to B'} = \big[[\vec{u}_1]_{B'} \quad [\vec{u}_2]_{B'} \quad \cdots \quad [\vec{u}_n]_{B'}\big]$ is the <u>transition matrix from $B$ to $B'$</u>.

The columns of the transition matrix are *the coordinate vectors of the old basis relative to the new basis.*

**Example 1.** Consider the bases $B = \{(1,0),(0,1)\}$ and $B' = \{(1,1),(1,2)\}$ for $\mathbb{R}^2$.

(a) Find the transition matrix $P_{B\to B'}$ from $B$ to $B'$.

$\vec{u}_1 = (1,0) = 2(1,1) - (1,2) = 2\vec{u}_1' - \vec{u}_2'$, so $[\vec{u}_1]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$\vec{u}_2 = (0,1) = -(1,1) + (1,2) = -\vec{u}_1' + \vec{u}_2'$, so $[\vec{u}_2]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Thus: $P_{B\to B'} = \big[[\vec{u}_1]_{B'} \;\; [\vec{u}_2]_{B'}\big] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

(b) Find the transition matrix $P_{B'\to B}$ from $B'$ to $B$.

$\vec{u}_1' = (1,1) = (1,0) + (0,1) = \vec{u}_1 + \vec{u}_2$, so $[\vec{u}_1']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\vec{u}_2' = (1,2) = \cdots = \vec{u}_1 + 2\vec{u}_2$, so $[\vec{u}_2']_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Thus: $P_{B'\to B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(c) Suppose that $[\vec{v}]_B = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Find $[\vec{v}]_{B'}$.

$$[\vec{v}]_{B'} = P_{B\to B'}[\vec{v}]_B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}.$$

Applying a change of basis from $B$ to $B'$ and then a change of basis from $B'$ to $B$ leaves coordinate vectors unchanged.

i.e. $$[\vec{v}]_B = P_{B'\to B}[\vec{v}]_{B'} = P_{B'\to B}\left(P_{B\to B'}[\vec{v}]_B\right)$$
$$= \left(P_{B'\to B}P_{B\to B'}\right)[\vec{v}]_B = I[\vec{v}]_B,$$

so $$P_{B'\to B}P_{B\to B'} = I.$$

This means that the transition matrices $P_{B\to B'}$ and $P_{B'\to B}$ are inverses of each other.

**Theorem.** If $P$ is the transition matrix from a basis $B$ to a basis $B'$ in the vector space $V$, then $P$ is invertible and $P^{-1}$ is the transition matrix from $B'$ to $B$.

We can find a transition matrix by row-reducing the matrix that has the vectors from each basis as columns.

$$\left[\underbrace{\vec{u}_1' \quad \vec{u}_2'}_{\substack{\text{new basis} \\ B'}} \;\middle|\; \underbrace{\vec{u}_1 \quad \vec{u}_2}_{\substack{\text{old basis} \\ B}}\right] \xrightarrow{\text{row operations}} \left[\, I \;\middle|\; P_{B\to B'} \right]$$

$$\xrightarrow{\text{row operations}} \left[\, P_{B'\to B} \;\middle|\; I \right]$$

**Example 2.** Find the transition matrix $P_{B\to B'}$ for the bases in Example 1.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_2\to R_2-R_1]{} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_1\to R_1-R_2]{} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Thus $$P_{B\to B'} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

**Theorem.** Let $B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be any basis for $\mathbb{R}^n$ and let $S = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then the transition matrix from $B$ to $S$ is

$$P_{B\to S} = [\vec{u}_1|\vec{u}_2|\cdots|\vec{u}_n].$$

In particular, if $A = [\vec{v}_1|\vec{v}_2|\cdots|\vec{v}_n]$ is an invertible matrix, then $A$ is a transition matrix from the basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ to the standard basis for $\mathbb{R}^n$.

## Section 4.8 Row Space, Column Space, Null Space

**Objectives.**

- Introduce the row space, column space, and null space for a matrix.
- Study how solutions to homogeneous and nonhomogeneous systems are related.
- Find a basis and the dimension of the row space, column space, and null space.

---

Given an $m \times n$ matrix $A$, we an define three natural subspaces of Euclidean space.

- the row space of $A$ is the set of all linear combinations of the row vectors of $A$

  **Question:** *Is the row space of $A$ a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - row vectors in $A$ have length $n$.

- the column space of $A$ is the set of all linear combinations of the column vectors of $A$

  **Question:** *Is the column space of $A$ a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - column vectors in $A$ have length $m$.

- the null space of $A$ is the set of all solutions to the equation $A\vec{x} = \vec{0}$

  **Question:** *Is the null space of $A$ a subspace of $\mathbb{R}^m$ or of $\mathbb{R}^n$?*

  - if $A\vec{x}$ is defined, then $\vec{x}$ has length $n$.

**Example 1.** Let $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ 1 & 3 \end{bmatrix}$.

(a) The set $\text{row}(A)$ (the row space of $A$) is a subspace of $\mathbb{R}^2$.

(b) Name one vector in $\text{row}(A)$. $\begin{bmatrix} 2 & 1 \end{bmatrix}$ (or $\begin{bmatrix} 4 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 \end{bmatrix}$, ...)

(c) The set $\text{col}(A)$ (the column space of $A$) is a subspace of $\mathbb{R}^3$.

(d) Name one vector in $\text{col}(A)$. $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$, ...)

(e) The set $\text{null}(A)$ (the null space of $A$) is a subspace of $\mathbb{R}^2$.

(f) Name one vector in $\text{null}(A)$. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (this is the only vector in $\text{null}(A)$).

**Example 2.** Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Is $(2, -2, 2)$ in $\text{row}(A)$? No.

if $k_1(1,0,-1) + k_2(0,1,1) = (2,-2,2)$, then $k_1 = 2$ and $k_2 = -2$, but $2(1,0,-1) + (-2)(0,1,1) = (2,-2,0) \neq (2,-2,2)$.

(b) What is a basis for $\text{row}(A)$?

$$S = \{(1,0,-1), (0,1,1)\}.$$

(c) What is the dimension of $\text{row}(A)$?

the dimension is the number of vectors in a basis.

$$\dim(\text{row}(A)) = 2.$$

(d) Is $(4, 2)$ in $\text{col}(A)$? Yes.

$$4\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}.$$

(e) What is a basis for $\text{col}(A)$?

$$S = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

note: $\begin{bmatrix}-1\\1\end{bmatrix} = -\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$, so the columns of $A$ are not linearly independent.

(f) What is the dimension of $\text{col}(A)$?

$$\dim(\text{col}(A)) = 2.$$

(g) Is $(2, -2, 2)$ in $\text{null}(A)$? Yes.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix}2\\-2\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

(h) What is a basis for $\text{null}(A)$?

$$S = \left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}.$$

note: every vector in $\text{null}(A)$ is a multiple of $\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

(i) What is the dimension of $\text{null}(A)$?

$$\dim(\text{null}(A)) = 1.$$

The column space of a matrix can also be described as the set of all vectors $\vec{b}$ in $\mathbb{R}^n$ for which the equation $A\vec{x} = \vec{b}$ has a solution.

**Theorem.** The equation $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in the column space of $A$.

**Example 3.** Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -3 \\ -2 \\ 6 \end{bmatrix}.$$

Show that $\vec{b}$ is in the column space of $A$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ -1 & 3 & 1 & -2 \\ 2 & 2 & 1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 6 & 3 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & -21 & 42 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Thus $3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 6 \end{bmatrix}$.

**Example 4.** Suppose that $\vec{x}_h$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$, and $\vec{x}_0$ is a solution of the nonhomogeneous system $A\vec{x} = \vec{b}$. Show that $\vec{x}_0 + k\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$ for all scalars $k$.

$A\vec{x}_0 = \vec{b}$ and $A\vec{x}_h = \vec{0}$, so

$$A(\vec{x}_0 + k\vec{x}_h) = A\vec{x}_0 + A(k\vec{x}_h) = \vec{b} + k(A\vec{x}_h) = \vec{b} + k(\vec{0}) = \vec{b}.$$

The importance of the last example is the following principle:

*The general solution for a consistent linear system is the sum of a particular solution for the linear system and the general solution for the corresponding homogeneous linear system.*

**Theorem.** Every solution $\vec{x}$ for a consistent linear system $A\vec{x} = \vec{b}$ can be written in the form

$$\vec{x} = \vec{x}_0 + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r,$$

where $\vec{x}_0$ is any solution for $A\vec{x} = \vec{b}$ and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is a basis for the null space of $A$.

**Finding a basis for the row space or column space of a matrix.**

Recall that two matrices are row equivalent if each can be obtained from the other through elementary row operations.

**Theorem.**

1. If $A$ and $B$ are row equivalent, then $\text{row}(A) = \text{row}(B)$.

2. If $A$ and $B$ are row equivalent, then $\text{null}(A) = \text{null}(B)$.

For a matrix $A$ in row-echelon form (such as in Example 2), identifying a basis for $\text{row}(A)$ or $\text{col}(A)$ is particularly easy – the row vectors containing a leading 1 form a basis for $\text{row}(A)$, and the column vectors containing a leading 1 form a basis for $\text{col}(A)$.

**Example 5.** Find a basis for $\text{row}(B)$ and a basis for $\text{col}(B)$ given that $B = \begin{bmatrix} 1 & -3 & 0 & 4 & -1 \\ 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

basis for $\text{row}(B)$: $S = \{(1,-3,0,4,-1), (0,1,2,-2,0), (0,0,0,1,1)\}$.

basis for $\text{col}(B)$: $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

More generally, a basis for $\text{row}(A)$ can be found by reducing $A$ to ref and applying the theorem above.

**Example 6.** Find a basis for $\text{row}(A)$ given that $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -4/3 \end{bmatrix}.$$

$S = \{(1,2,-1,3), (0,1,2,-3), (0,0,1,-4/3)\}$ is a basis for $\text{row}(A)$.

The next theorem allows us to find a basis for $\mathrm{col}(A)$ – more specifically, a basis for $\mathrm{col}(A)$ that consists entirely of columns of $A$.

**Theorem.** Suppose that $A$ and $B$ are row equivalent.

1. If a set of columns of $A$ are linearly independent, then the corresponding columns of $B$ are also linearly independent.
2. If a set of columns of $A$ are a basis for $\mathrm{col}(A)$, then the corresponding columns of $B$ are a basis for $\mathrm{col}(B)$.

**Example 7.** Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$.

(a) Find a matrix $B$ in row-echelon form that is row equivalent to $A$.

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & -1 & 5 & -3 & -10 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & -1 & 5 & -3 & -10 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 0 & 1 & -5 & 3 & 10 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Identify a basis for $\mathrm{col}(B)$ in part (a).

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix} \right\}.$$

the columns of $B$ that contain a "leading 1" form a basis for $\mathrm{col}(B)$.

(c) Use the theorem above to identify a basis for $\mathrm{col}(A)$ that consists entirely of columns of $A$.

- because columns 1,2,5 are a basis for $\mathrm{col}(B)$, the corresponding columns of $A$ are a basis for $\mathrm{col}(A)$.

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

(d) What is the dimension of $\mathrm{col}(A)$?

$$\dim(\mathrm{col}(A)) = 3.$$

Suppose that we want to find a basis for $\mathrm{row}(A)$ that consists entirely of rows of $A$. One way to do this is to apply the method from the previous page to the matrix $A^T$. This gives a basis for $\mathrm{col}(A^T)$ that consists of columns of $A^T$ – transposing this basis gives a basis for $\mathrm{row}(A)$ that consists of rows of $A$.

**Example 8.** Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 & 1 & 4 \\ 3 & 2 & -1 & 0 & 2 \\ 0 & -1 & 5 & -3 & -2 \end{bmatrix}$ from Example 7.

(a) Find a basis for $\mathrm{col}(A^T)$ that consists entirely of columns of $A^T$.

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & -1 & 5 \\ 1 & 0 & -3 \\ 4 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 0 & 5 & 5 \\ 0 & -3 & -3 \\ 0 & -10 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \\ 0 & -3 & -3 \\ 0 & -10 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because all three columns of the reduced matrix contain a leading 1, we need all three columns of $A^T$ in a basis for $\mathrm{col}(A^T)$.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 5 \\ -3 \\ -2 \end{bmatrix} \right\}.$$

(b) Find a basis for $\mathrm{row}(A)$ that consists entirely of rows of $A$.

$$S = \{ (1,1,-2,1,4), (3,2,-1,0,2), (0,-1,5,-3,-2) \}.$$

(c) What is the dimension of $\mathrm{row}(A)$? $\dim(\mathrm{row}(A)) = 3.$

## Section 4.9 Rank, Nullity, and the Fundamental Matrix Spaces

**Objectives.**

- Define the rank and nullity of a matrix, and see how these are related.
- Introduce the orthogonal complement of a subspace.
- Extend the Equivalence Theorem.

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Recall the following definitions from Section 4.8.

- the row space of $A$ is the set of all linear combinations of the row vectors of $A$
- the column space of $A$ is the set of all linear combinations of the column vectors of $A$
- the null space of $A$ is the set of all solutions to the equation $A\vec{x} = \vec{0}$

The dimensions of these three spaces are related, and depend on the number of "leading variables" and "free variables" in a linear system.

**Theorem.** The row space and column space of a matrix $A$ have the same dimension.

The common dimension of the row space and the column space of $A$ is called the rank of $A$. The dimension of the null space of a matrix $A$ is called the nullity of $A$.

**Example 1.** What is the rank of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$? What is the nullity of $A$?

$\{(1,0),(0,1)\}$ is a basis for row$(A)$, so rank$(A) = 2$.

(also, $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for col$(A)$.)

The only vector in null$(A)$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so nullity$(A) = 0$.

i.e. null$(A)$ is the zero vector space.

**Theorem.** If $A$ is an $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$.

number of columns.

We can also relate the rank and nullity of a matrix with the number of leading variables and the number of free variables in a homogeneous linear system.

**Theorem.** Let $A$ be an $m \times n$ matrix. Then $\text{rank}(A)$ is the number of leading variables in the general solution to $A\vec{x} = \vec{0}$, and $\text{nullity}(A)$ is the number of free variables in the general solution to $A\vec{x} = \vec{0}$.

**Example 2.** The matrices $A$, $B$, and $C$ below are row equivalent.

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 0 & 2 \\ 2 & 4 & 2 & 1 & 5 \\ 1 & 0 & 3 & -2 & -2 \end{bmatrix} \xrightarrow[\text{ref}]{} B = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{rref}]{} C = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3 = s$ $\quad x_5 = t$

(a) Find a basis for row$(A)$.

• row$(A)$ = row$(B)$ = row$(C)$ because $A, B, C$ are row equivalent.

basis for row$(A) = \{(1,1,2,-1,0), (0,1,-1,1,2), (0,0,0,1,1)\}$.

(b) Find a basis for col$(A)$.

$$\text{basis for col}(B) = \left\{ \begin{array}{c} \text{col } 1 \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} \text{col } 2 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} \text{col } 4 \\ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{array} \right\}, \text{ so}$$

use the corresponding columns of $A$.

$$\text{basis for col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

(c) What is the rank of $A$?

rank$(A) = 3$

because ... dim(row$(A)$) = 3 or dim(col$(A)$) = 3 or $A$ has 3 leading variables.

(d) Find a basis for null$(A)$.

• sol$^{\underline{n}}$ to $A\vec{x} = \vec{0}$ is $x_3 = s$, $x_5 = t$, $x_1 = -3s$, $x_2 = s - t$, $x_4 = -t$,

or $\vec{x} = (-3s, s-t, s, -t, t) = s(-3,1,1,0,0) + t(0,-1,0,-1,1)$.

basis for null$(A) = \{(-3,1,1,0,0), (0,-1,0,-1,1)\}$.

(e) What is the nullity of $A$?

nullity$(A) = 2$

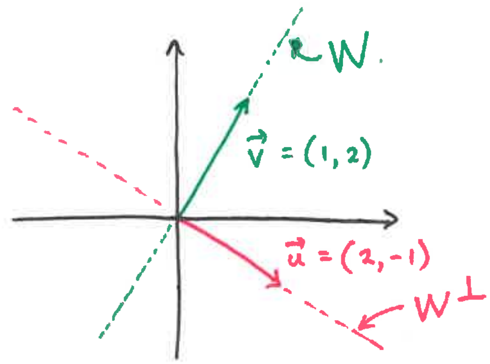
because ... dim(null$(A)$) = 2 or $A$ has 2 free variables or $n - \text{rank}(A) = 5 - 3 = \underline{2}$.

If $W$ is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to *every* vector in $W$ is called the orthogonal complement of $W$, and is denoted by $W^\perp$. $\leftarrow$ "W perp"

**Example 3.** Let $W = \text{span}\{(1,2)\}$, which is a subspace of $\mathbb{R}^2$.

(a) Find a vector in $W^\perp$. $\vec{u} = (2,-1)$.

(why? if $\vec{v}$ is in $W$, then $\vec{v} = k(1,2) = (k,2k)$.
Thus $(2,-1)\cdot(k,2k) = 2k - 2k = 0$.)




(b) Describe the set of all vectors in $W^\perp$.

$W^\perp$ contains all vector parallel to $(2,-1)$.
(why? $(2\ell,-\ell)\cdot(k,2k) = 2k\ell - 2k\ell = 0$.)

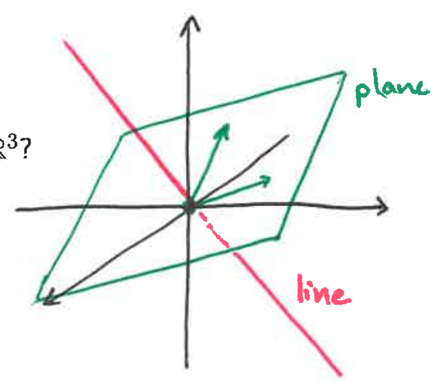
note: $\{\vec{0}\}$ is the orthogonal complement of $\mathbb{R}^2$ in $\mathbb{R}^2$.

**Theorem.** If $W$ is a subspace of $\mathbb{R}^n$, then:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. The only vector in both $W$ and $W^\perp$ is $\vec{0}$.
3. The orthogonal complement of $W^\perp$ is $W$.

**Example 4.** (a) What is the orthogonal complement of a line through the origin in $\mathbb{R}^3$?

a plane through the origin.
(i.e. the plane that is orthogonal
to any vector on the line)




(b) What is the orthogonal complement of a plane through the origin in $\mathbb{R}^3$?

a line through the origin.
(i.e. the line that is orthogonal
to any vector on the plane)

Recall that if $\vec{x}_h$ is a solution to the homogeneous linear system $A\vec{x} = \vec{0}$, then $\vec{x}_h$ is orthogonal to every row of $A$. That is, $\vec{x}_h \cdot \vec{r}_i = 0$ where $\vec{r}_i$ is the $i$th row of $A$.

**Theorem.** If $A$ is an $m \times n$ matrix, then:

1. The null space of $A$ and the row space of $A$ are orthogonal complements in $\mathbb{R}^n$.
2. The null space of $A^T$ and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$.

**Example 5.** Let $\vec{x}_h$ be a solution to the homogeneous linear system $A\vec{x} = \vec{0}$, and let $\vec{r}$ be a vector in the row space of $A$. Show that $\vec{x}_h$ is orthogonal to $\vec{r}$.

Because $\vec{r}$ is in the row space of $A$, we can write

$$\vec{r} = c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_m\vec{r}_m$$

where $\vec{r}_i$ is the $i^{th}$ row of $A$.

Then:

$$\begin{aligned}
\vec{x}_h \cdot \vec{r} &= \vec{x}_h \cdot (c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots + c_m\vec{r}_m) \\
&= \vec{x}_h \cdot (c_1\vec{r}_1) + \vec{x}_h \cdot (c_2\vec{r}_2) + \cdots + \vec{x}_h \cdot (c_m\vec{r}_m) \\
&= c_1\, \vec{x}_h \cdot \vec{r}_1 + c_2\, \vec{x}_h \cdot \vec{r}_2 + \cdots + c_m\, \vec{x}_h \cdot \vec{r}_m \\
&= c_1(0) + c_2(0) + \cdots + c_m(0) \\
&= 0.
\end{aligned}$$

That is, $\vec{x}_h$ is orthogonal to $\vec{r}$.

<u>note:</u> This proves part (1) of the theorem above, because we have shown that any vector in null$(A)$ is orthogonal to any vector in row$(A)$.

We finally have all the ingredients to state the "Equivalence Theorem" in full.

**Equivalence Theorem.** If $A$ is an $n \times n$ matrix with no repeated rows or repeated columns, then the following statements are equivalent.

1. $A$ is invertible.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ can be written as a product of elementary matrices.
5. $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
6. $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ vector $\vec{b}$.
7. $\det A \neq 0$.
8. The column vectors of $A$ are linearly independent.
9. The row vectors of $A$ are linearly independent.
10. The column vectors of $A$ span $\mathbb{R}^n$.
11. The row vectors of $A$ span $\mathbb{R}^n$.
12. The column vectors of $A$ are a basis for $\mathbb{R}^n$.
13. The row vectors of $A$ are a basis for $\mathbb{R}^n$.
14. $\text{rank}(A) = n$.
15. $\text{nullity}(A) = 0$.
16. The orthogonal complement of $\text{null}(A)$ is $\mathbb{R}^n$.
17. The orthogonal complement of $\text{row}(A)$ is $\{\vec{0}\}$.

## Section 5.1 Eigenvalues and Eigenvectors

"eigen" = "own"

**Objectives.**

- Introduce eigenvalues and eigenvectors for a matrix or matrix transformation.
- Find eigenvalues, eigenvectors, and eigenspaces.

---

Suppose that $\vec{x}$ is a non-zero vector and $A$ is a square matrix. If $A\vec{x} = \lambda\vec{x}$ for some scalar $\lambda$, then $\lambda$ is an eigenvalue of $A$ and $\vec{x}$ is an eigenvector of $A$ corresponding to $\lambda$.

**Example 1.** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Compute $A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Which of these vectors is an eigenvector of $A$? What are the eigenvalues?

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ i.e. $\lambda = 1$.

Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $A$ with eigenvalue $\lambda = 1$.

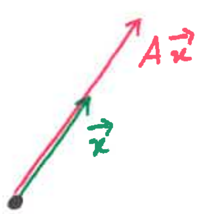
$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ i.e. $\lambda = 2$.

Thus $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ with eigenvalue $\lambda = 2$.

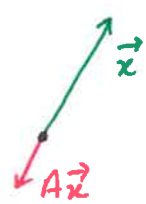
$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \lambda\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an eigenvector of $A$.

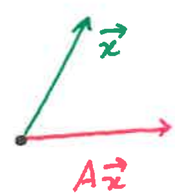
Loosely speaking, an eigenvector of an $n \times n$ matrix $A$ (or of the matrix operator $T_A$) is a direction in $\mathbb{R}^n$ that is unchanged when multiplying by $A$. That is, $\vec{x} \neq \vec{0}$ is an eigenvector of $A$ if $\vec{x}$ and $A\vec{x}$ are parallel.




$\vec{x}$ is an eigenvector of $A$ with $\lambda > 1$.




$\vec{x}$ is an eigenvector of $A$ with $-1 < \lambda < 0$.




$\vec{x}$ is not an eigenvector of $A$.

(b/c $A\vec{x} \neq \lambda\vec{x}$)

characteristic equation of $A$.

**Theorem.** If $A$ is a square matrix, then $\lambda$ is an eigenvalue of $A$ if and only if $\det(\lambda I - A) = 0$.

**Proof.** Suppose $\lambda$ is an eigenvalue of $A$. Then there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. That is, $A\vec{x} = \lambda I\vec{x}$, so $\vec{0} = \lambda I\vec{x} - A\vec{x} = (\lambda I - A)\vec{x}$. Thus $\det(\lambda I - A) = 0$.

Suppose $\det(\lambda I - A) = 0$. Then there is a nonzero vector $\vec{x}$ such that $(\lambda I - A)\vec{x} = \vec{0}$. Thus $\lambda I\vec{x} - A\vec{x} = \vec{0}$, so

$A\vec{x} = \lambda I\vec{x} = \lambda\vec{x}$. Therefore, $\lambda$ is an eigenvalue of $A$.

**Example 2.** Use the theorem above to find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-2 \end{bmatrix}.$$

characteristic polynomial of $A$.

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-2 \end{bmatrix} = (\lambda-1)(\lambda-2) - (-1)(0) = (\lambda-1)(\lambda-2).$$

- solve $\det(\lambda I - A) = 0$:

  $(\lambda-1)(\lambda-2) = 0 \implies \lambda = 1, 2$.

The eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$.

**Strategy.** To find the eigenvalues of $A$:

- set up the characteristic equation/polynomial of $A$
- find all the solutions of the characteristic equations.

**Example 3.** Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$.

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-1 & -2 & 0 \\ -3 & \lambda-1 & -2 \\ 0 & -3 & \lambda-1 \end{bmatrix} = (\lambda-1)\left((\lambda-1)^2-6\right) - (-2)\left((-3)(\lambda-1)\right)$$

$$= (\lambda-1)\left((\lambda-1)^2-6-6\right) = \underbrace{(\lambda-1)(\lambda^2-2\lambda-11)}_{\text{characteristic polynomial}}.$$

$\det(\lambda I - A) = 0 \implies \lambda = 1$ or $\lambda^2 - 2\lambda - 11 = 0 \Leftarrow$ use quadratic formula!!!

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4+44}}{2} = 1 \pm \frac{\sqrt{48}}{2} = 1 \pm 2\sqrt{3}.$$

The eigenvalues are $\lambda = 1$, $\lambda = 1 + 2\sqrt{3}$, $\lambda = 1 - 2\sqrt{3}$.

The eigenvalues of a triangular matrix can be found 'by inspection' (that is, without solving the characteristic polynomial).

**Theorem.** If $A$ is triangular, then the eigenvalues of $A$ are the entries on the main diagonal.

**Example 4.** Find the eigenvalues of each matrix.

$$\begin{bmatrix} 3 & 9 & -4 \\ 0 & -7 & 5 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

$\lambda = 3, -7, 4$. $\qquad \lambda = \frac{1}{2}, \frac{3}{2}, 2$. $\qquad \lambda = a, b, c, d$.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-3 & -9 & 4 \\ 0 & \lambda+7 & -5 \\ 0 & 0 & \lambda-4 \end{bmatrix} = (\lambda-3)(\lambda+7)(\lambda-4).$$

**Theorem.** If $A$ is a square matrix, then the following statements are equivalent.

1. $\lambda$ is an eigenvalue of $A$.
2. $\lambda$ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
3. The system $(\lambda I - A)\vec{x} = \vec{0}$ has nontrivial solutions.
4. There is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.

Now that we know how to find eigenvalues for a matrix, we turn our attention to finding the eigenvectors corresponding to each eigenvalue. If $\lambda$ is an eigenvalue of $A$, then the eigenvectors corresponding to $\lambda$ are the nonzero vectors $\vec{x}$ such that $(\lambda I - A)\vec{x} = \vec{0}$. This solution space is the eigenspace corresponding to $\lambda$.

- find all eigenvalues of $A$
- solve $(\lambda I - A)\vec{x} = \vec{0}$ for each eigenvalue $\lambda$.

**Example 5.** Find the eigenspaces of the matrix $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix} = (\lambda+1)\lambda - 6 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2).$$

The eigenvalues of $A$ are $\lambda = -3$, $\lambda = 2$.

$\lambda = 2$: $\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\Longrightarrow x_1 = t,\ x_2 = t.$ use "elimination" to solve.

Thus $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda = 2$.

$\lambda = -3$: $\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Longrightarrow x_1 = -\frac{3}{2}t,\ x_2 = t.$

Thus $\left\{\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda = -3$.

cofactor expansion!!!

**Example 6.** Find the eigenspaces of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} = \cdots = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda-1)(\lambda-2)^2.$$

The eigenvalues of $A$ are $\lambda=1$ and $\lambda=2$. ← repeated eigenvalue.

$\underline{\lambda=1:}$ $\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -2s,\ x_2 = s,\ x_3 = s.$

The ~~eigenvalue~~ eigenvectors for $\lambda=1$ are $s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, so $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for this eigenspace.

$\underline{\lambda=2:}$ $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -t,\ x_2 = s,\ x_3 = t$

The eigenvectors for $\lambda=2$ are $\begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for this eigenspace.

**Theorem.** The square matrix $A$ is invertible if and only if $\lambda = 0$ is not an eigenvalue of $A$.

**Example 7.** Find the eigenvalues of $A = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$.

$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & 3 \\ 0 & \lambda \end{bmatrix} = (\lambda-1)\lambda$, so $\lambda = 0, 1$ are the eigenvalues of $A$. Thus $A$ is <u>not</u> invertible.

## Section 5.2 Diagonalization

**Objectives.**

- Define similarity transformations and identify some properties of similar matrices.
- Introduce the idea of diagonalizing a matrix.
- Use diagonalization to compute powers of a matrix efficiently.

---

Let $A$ and $P$ be $n \times n$ matrices with $P$ invertible. The transformation that sends $A$ to the matrix product $P^{-1}AP$ is called a similarity transformation.

More generally, if $A$ and $B$ are $n \times n$ matrices then we say that $B$ is similar to $A$ if there is an invertible matrix $P$ such that $B = P^{-1}AP$.

**Example 1.** Suppose that $B$ is similar to $A$. Show that $A$ is similar to $B$.

Because $B$ is similar to $A$, there is an invertible matrix $P$ such that $B = P^{-1}AP$.

Then $PBP^{-1} = P(P^{-1}AP)P^{-1} = (PP^{-1})A(PP^{-1}) = IAI = A$.

That is $A = Q^{-1}BQ$ where $Q = P^{-1}$. Thus $A$ is similar to $B$.

(Notice that the previous example allows us to say that $A$ and $B$ are similar if one is similar to the other.)

Similar matrices share several important properties. In particular, if $A$ and $B$ are similar then $A$ and $B$ have the same ...

determinant, rank, nullity, trace, characteristic polynomial, eigenvalues, ...

note: similar matrices represent the same linear transformation with respect to different bases.

**Example 2.** Suppose that $A$ and $B$ are similar matrices. Show that $\det(A) = \det(B)$.

Because $A$ and $B$ are similar, there is an invertible matrix $P$ such that $B = P^{-1}AP$.

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$
$$= \frac{1}{\det(P)}\det(A)\det(P) = \det(A).$$

An $n \times n$ matrix $A$ is diagonalizable if it is similar to a diagonal matrix. That is, if there is an invertible matrix $P$ such that $P^{-1}AP$ is diagonal, in which case we say that $P$ diagonalizes $A$.

**Example 3.** Consider the $2 \times 2$ matrices $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Show that $P$ diagonalizes $A$.

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}.$$

Thus $P$ ~~diagonal~~ diagonalizes $A$.

(b) What are the eigenvalues of $A$?

$P^{-1}AP$ has eigenvalues $\lambda = 4, 5$, so $A$ also has eigenvalues $\lambda = 4, 5$.

The key ingredient for diagonalizing a matrix is the set of eigenvectors of the matrix.

**Theorem.** An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

**Theorem.** If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix $A$, and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are corresponding eigenvectors, then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

It follows from the previous two theorems that an $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

Why? each eigenvalue corresponds to (at least) one eigenvector, so $n$ distinct eigenvalues gives us $n$ linearly independent eigenvectors.

Thus these $n$ eigenvectors are a basis for $\mathbb{R}^n$.

**Strategy.** To find a matrix that diagonalizes $A$:

- find the eigenvalues and corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$ of $A$.
- if you find $n$ eigenvectors, then $P = [\vec{v}_1 | \vec{v}_2 | \cdots | \vec{v}_n]$ diagonalizes $A$.

**Example 4.** Find a matrix $P$ that diagonalizes $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

From Ex. 6, Section 5.1, the eigenvalues of $A$ are $\lambda = 1$ (with eigenvector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$) and $\lambda = 2$ (with eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$). These three eigenvectors are linearly independent, so $P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalizes $A$.

eigenvalues of $A$!!

check: $P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

note: if we chose $P = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, then $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**Example 5.** Show that the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ is not diagonalizable.

$A$ is triangular, so the eigenvalues are $\lambda = 1$ (repeated) and $\lambda = 4$.

$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda-4 \end{bmatrix} = (\lambda-1)^2(\lambda-4)$. $\Rightarrow \lambda = 1, \lambda = 4$.

$\lambda = 1$: $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = t,\ x_2 = 0,\ x_3 = 0$.

The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for this eigenspace.

$\lambda = 4$: $\begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = t,\ x_2 = 3t,\ x_3 = 9t$.

The set $\left\{ \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \right\}$ is a basis for the eigenspace.

Because $A$ has only two linearly independent eigenvectors, we cannot diagonalize $A$.

**Example 6.** Explain why the matrix $A = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ is diagonalizable.

The eigenvalues of $A$ are $\lambda = 1, 2, 4, 5$. These are distinct, so $A$ has four linearly independent eigenvectors. Therefore $A$ is diagonalizable.

One application of diagonalization is finding powers of a matrix. Recall that if $D$ is a diagonal matrix, then $D^k$ can be found by raising each diagonal entry to the power $k$.

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix} \quad (\text{for } k > 0).$$

Suppose that $A$ is similar to a diagonal matrix $D$, so that $A = P^{-1}DP$ where $P$ is invertible. Then:

i.e. $A$ is diagonalizable.

$$A^k = (P^{-1}DP)^k = (P^{-1}DP)(P^{-1}DP)\cdots(P^{-1}DP)$$
$$= P^{-1}D(PP^{-1})D(PP^{-1})D\cdots DP = P^{-1}D^kP.$$

**Example 7.** Compute $A^5$ for the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ in Example 4.

$A = PDP^{-1}$ where $P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Thus: $A^5 = (PDP^{-1})^5 = PD^5P^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$$= \cdots = \begin{bmatrix} -30 & 0 & -62 \\ 31 & 32 & 31 \\ 31 & 0 & 63 \end{bmatrix}.$$