MATH 241 Calculus III
Chapter 14 Review
Multidimensional Differentiation

14.1 Functions of Several Variables

Find the specific function value.

1) Find f(3, 0, 9) when $f(x, y, z) = 3x^2 + 3y^2 - z^2$.

Find the domain and range and describe the level curves for the function f(x,y).

2) $f(x, y) = \dfrac{1}{3x^2 + 10y^2}$

3) $f(x, y) = \sqrt{25 - x^2 - y^2}$

Find the equation for the level surface of the function through the given point.

4) $f(x, y, z) = \dfrac{x^2 y}{xz + y^2}$, (2, 7, 8)

14.2 Limits and Continuity

Find the limit.

5) $\displaystyle\lim_{(x, y) \to (0, 0)} \sin\left(\frac{x^3 + y^{10}}{x - y + 7}\right)$

6) $\displaystyle\lim_{\substack{(x, y) \to (9, 4) \\ y \neq 4}} \frac{xy + 4y - 4x - 16}{y - 4}$

7) $\displaystyle\lim_{\substack{(x, y) \to \left(\frac{25}{2}, \frac{25}{2}\right) \\ x + y \neq 25}} \frac{x + y - 25}{\sqrt{x + y} - 5}$

At what points is the given function continuous?

8) $f(x, y) = \dfrac{xy}{x + y}$

Determine whether the statement is true or false.

9) If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b), then $\displaystyle\lim_{(x, y) \to (a, b)} f(x, y) = L$.

14.3 Partial Derivatives

Find all the first order partial derivatives for the following function.

10) $f(x, y) = (3x^4y^5 + 7)^2$

Find all the first order partial derivatives for the following function.

11) $f(x, y, z) = xz\sqrt{x + y}$

Provide an appropriate response.

12) Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then what does Clairaut's Theorem conclude?

13) Write the limit definition for $f_x(a, b)$ and $f_y(a, b)$.

Determine whether the statement is true or false.

14) $f_y(a, b) = \lim\limits_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$

15) $f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y}$

Solve the problem.

16) The Van der Waals equation provides an approximate model for the behavior of real gases. The equation is $P(V, T) = \dfrac{RT}{V - b} - \dfrac{a}{V^2}$, where P is pressure, V is volume, T is Kelvin temperature, and a,b , and R are constants. Find the partial derivative of the function with respect to each variable.

Find all the second order partial derivatives of the given function.

17) $f(x, y) = \cos(xy^2)$

14.4 Tangent Planes and Linear Approximations

Solve the problem.

18) Find the equation for the tangent plane to the surface $z = 2x^2 - 4y^2$ at the point $(2, 1, 4)$.

Find the linear approximation of the function at the given point.

19) $f(x, y) = 7x^2 - 3y^2 - 5$ at $(-2, 2)$

14.5 Chain Rule

Solve the problem.

20) Evaluate $\frac{dw}{dt}$ at $t = \frac{1}{2}\pi$ for the function $w(x, y) = x^2 - y^2 + 8x$; $x = \cos t$, $y = \sin t$.

Solve the problem.

21) Evaluate $\frac{\partial u}{\partial x}$ at $(x, y, z) = (1, 2, 1)$ for the function $u(p, q, r) = p^2 - q^2 - r$; $p = xy$, $q = y^2$, $r = xz$.

Write a chain rule formula for the following derivative.

22) $\frac{dw}{dt}$ for $w = f(p, q, r)$; $p = g(t)$, $q = h(t)$, $r = k(t)$

23) $\frac{\partial u}{\partial x}$ for $u = f(p, q)$; $p = g(x, y, z)$, $q = h(x, y, z)$

Use implicit differentiation to find the specified derivative at the given point.

24) Find $\frac{dy}{dx}$ at the point (2, 1) for $\ln x + xy^2 + \ln y = 0$.

25) Find $\frac{\partial y}{\partial z}$ at the point (1, 3, 2) for $\frac{5}{x^2} + \frac{1}{y^2} + \frac{3}{z^2} = 0$.

Solve the problem.

26) A simple electrical circuit consists of a resistor connected between the terminals of a battery. The voltage V (in volts) is dropping as the battery wears out. At the same time, the resistance R (in ohms) is increasing as the resistor heats up. The power P (in watts) dissipated by the circuit is given by $P = \frac{V^2}{R}$. Use the equation

$$\frac{dP}{dt} = \frac{\partial P}{\partial V}\frac{dV}{dt} + \frac{\partial P}{\partial R}\frac{dR}{dt}$$

to find how much the power is changing at the instant when R = 2 ohms, V = 2 volts, dR/dt = 0.05 ohms/sec and dV/dt = -0.04 volts/sec.

14.6 Directional Derivatives and Gradient Vector

Compute the gradient of the function at the given point.

27) $f(x, y) = \ln(10x - 9y)$, (8, 4)

Find the derivative of the function at the given point in the direction of A.

28) $f(x, y) = 4x^2 - 3y$, (-4, -10), $A = 3i - 4j$

Provide an appropriate response.

29) Find the unit vector in the direction that the function is increasing most rapidly at the point P_0.
$f(x, y, z) = xy - \ln(z)$, $P_0(1, 2, 2)$

30) Write the limit definition of the directional derivative of f at (x_0, y_0) in the direction of unit vector $u = \langle a, b\rangle$.

Solve the problem.

31) Find the derivative of the function $f(x, y) = \tan^{-1} \frac{y}{x}$ at the point $(6, -6)$ in the direction in which the function increases most rapidly.

32) Find the derivative of the function $f(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ at the point $(-5, 5, -5)$ in the direction in which the function decreases most rapidly.

33) Find the equations of the tangent plane and the normal line to $xy^2z^3 = 8$ at $(2, 2, 1)$.

1) -54
2) Domain: all points in the xy-plane except (0, 0); range: real numbers > 0; level curves: ellipses $3x^2 + 10y^2 = c$
3) Domain: all points in the xy-plane satisfying $x^2 + y^2 \le 25$; range: real numbers $0 \le z \le 5$; level curves: circles with centers at (0, 0) and radii r, $0 < r \le 5$
4) $\frac{28}{65} = \frac{x^2y}{xz + y^2}$
5) 0
6) 13
7) 10
8) All (x, y) such that $x \ne -y$
9) False
10) $\frac{\partial f}{\partial x} = 24x^3y^5(3x^4y^5 + 7)$; $\frac{\partial f}{\partial y} = 30x^4y^4(3x^4y^5 + 7)$
11) $\frac{\partial f}{\partial x} = z\left(\sqrt{x + y} + \frac{x}{2\sqrt{x + y}}\right)$; $\frac{\partial f}{\partial y} = \frac{xz}{2\sqrt{x + y}}$; $\frac{\partial f}{\partial z} = x\sqrt{x + y}$
12) $f_{xy}(a,b) = f_{yx}(a,b)$
13) $f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$, $f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$
14) True
15) False
16) $P_V = \frac{2a}{V^3} - \frac{RT}{(V - b)^2}$; $P_T = \frac{R}{V - b}$
17) $\frac{\partial^2 f}{\partial x^2} = -y^4 \cos xy^2$; $\frac{\partial^2 f}{\partial y^2} = -2x[2xy^2 \cos(xy^2) + \sin(xy^2)]$; $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} =$
$-2y[xy^2 \cos(xy^2) + \sin(xy^2)]$;
18) $8x - 8y - z = 4$
19) $L(x, y) = -28x - 12y - 21$
20) -8
21) 7
22) $\frac{dw}{dt} = \frac{\partial w}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial q}\frac{dq}{dt} + \frac{\partial w}{\partial r}\frac{dr}{dt}$
23) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x}$
24) $-\frac{3}{10}$
25) $-\frac{81}{8}$
26) -0.13 watts

27) $\frac{5}{22}\mathbf{i} - \frac{9}{44}\mathbf{j}$

28) $-\frac{84}{5}$

29) $\frac{1}{\sqrt{21}}(4\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

30) $D_{\mathbf{u}}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$

31) $\frac{\sqrt{2}}{12}$

32) $-\frac{2}{5}\sqrt{2}$

33) $x + 2y + 6z = 12$; $x - 2 = \frac{y - 2}{2} = \frac{z - 1}{6}$