Tues 11/25

§16.7 Surface Integrals

- $f(x,y,z)$ multidimensional scalar function
- $\vec{F}(x,y,z)$ 3-D vector field.

For $f(x,y,z)$, the surface integral over S is

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

Example: Evaluate $\iint_S y\,dS$, S is the surface
$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle$,
$0 \le u \le 1$, $0 \le v \le \pi$.

Solution: $f(x,yz) = y = u\sin v$

$$\vec{r}_u = \langle \cos v,\ \sin v,\ 0\rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= \hat{i}(\sin v - 0) - \hat{j}(\cos v - 0) + \hat{k}(u\cos^2 v + u\sin^2 v)$$

$$= \langle \sin v,\ -\cos v,\ u\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{u^2+1}$$

$$\iint_S y\,dS = \iint_D u\sin v\sqrt{u^2+1}\,dA$$

$$= \int_0^\pi \int_0^1 u\sqrt{u^2+1}\,\sin v\,du\,dv$$

$$= \int_0^1 u\sqrt{u^2+1}\,du \int_0^\pi \sin v\,dv$$

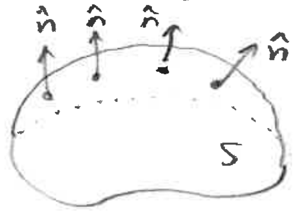
$$= \frac{2}{3}\left(\frac{1}{2}\right)(u^2+1)^{3/2}\Big|_0^1 \ (-\cos v)\Big|_0^\pi$$

$$= \frac{1}{3}(2\sqrt{2}-1)(1+1) = \frac{2}{3}(2\sqrt{2}-1)$$

Surface Integral of Vector Field

We saw that $\vec{r}_u \times \vec{r}_v = \vec{n}$ normal to the surface.

Also $-\vec{n} = \vec{r}_v \times \vec{r}_u$



Oriented surfaces have boundaries so that there are two surfaces on opposite sides.

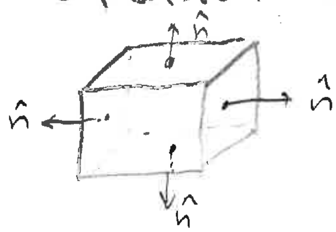
This is one orientation.

Another orientation is

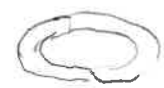
A unit normal vector $\hat{n}$ is

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

Typically, we take outward normals to be positive. So for a closed surface,



Every surface is orientable if it is not a Mobius strip.



For an oriented surface, we write

$$d\vec{S} = \hat{n}\, dS = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dS = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= (\vec{r}_u \times \vec{r}_v)\, dA$$

So for a vector field $\vec{F}(x,y,z)$ continuous on an oriented surface S, the surface integral over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

↑ If this is opposite direction, multiply by -1.

Example: Find $\iint_S \vec{F} \cdot d\vec{S}$ for $\vec{F}(x,y,z) = 7e^{xy}\hat{\imath} - 37e^{xy}\hat{\jmath} + xy\hat{k}$ and S is the parallelogram $x = u+v$, $y = u-v$, $z = 1+2u+v$, $0 \le u \le 2$, $0 \le v \le 1$, with upward orientation.

Solution: $\vec{r}(u,v) = \langle u+v, u-v, 1+2u+v \rangle$

$\vec{r}_u = \langle 1, 1, 2 \rangle$

$\vec{r}_v = \langle 1, -1, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 3, 1, -2 \rangle$

downward

$\vec{n} = -(\vec{r}_u \times \vec{r}_v) = \langle -3, -1, 2 \rangle$

upward orientation

$$\vec{F} \cdot \vec{n} = -37e^{xy} + 37e^{xy} + 2xy = 2xy$$
$$= 2(u+v)(u-v) = 2(u^2 - v^2)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D 2(u^2 - v^2)\,dA = 2\int_0^1 \int_0^2 (u^2 - v^2)\,du\,dv$$
$$= 2\int_0^1 \left[\frac{u^3}{3} - uv^2\right]_0^2 dv = 2\int_0^1 \left(\frac{8}{3} - 2v^2\right) dv$$
$$= 2\left[\frac{8}{3}v - \frac{2}{3}v^3\right]_0^1 = 2\left(\frac{8}{3} - \frac{2}{3}\right) = 4$$