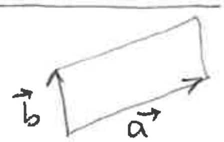
Mon 11/24

## §16.6 Surface Areas

**Plane Surface**

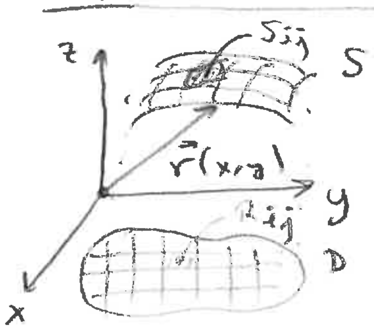



$$\vec{n} = \vec{a} \times \vec{b} = \langle n_1, n_2, n_3 \rangle$$

$$n_1 x + n_2 y + n_3 z = n_1 x_0 + n_2 y_0 + n_3 z_0$$

$$|\vec{n}| = |\vec{a} \times \vec{b}| = \text{area of parallelogram}$$

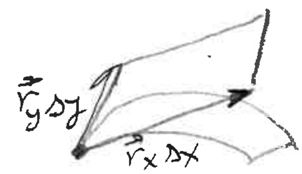
**Graph of a Function Surface**




$\Delta R_{ij} = \Delta x \Delta y = \Delta A$

$z = f(x,y)$

$\Delta S_{ij}$




$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

$$\vec{r}_x = \langle 1, 0, \frac{\partial f}{\partial x} \rangle$$

$$\vec{r}_y = \langle 0, 1, \frac{\partial f}{\partial y} \rangle$$

$$\Delta S_{ij} \approx |\vec{r}_x \times \vec{r}_y| \Delta x \Delta y$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = \hat{i}\left(0 - \frac{\partial z}{\partial x}\right) - \hat{j}\left(\frac{\partial z}{\partial y} - 0\right) + \hat{k}(1-0)$$

$$= \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

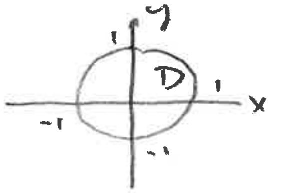
Taking the limit as subrectangles $R_{ij}$ go to $\infty$,

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$A(S) = \iint_S dS = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example: Find the area of the surface $z = xy$ that lies within the cylinder $x^2+y^2=1$.

Solution: $A(S) = \iint_S dS = \iint_D \sqrt{1+y^2+x^2}\, dA$




Let $x = r\cos\theta$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$A(S) = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, r\,dr\,d\theta$$

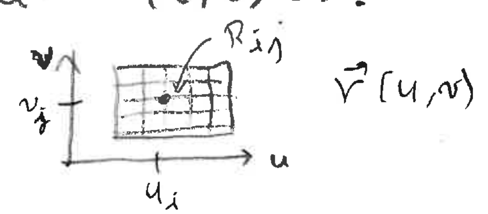
$$= \int_0^{2\pi} d\theta \int_0^1 \frac{2r}{2}\sqrt{1+r^2}\, dr$$

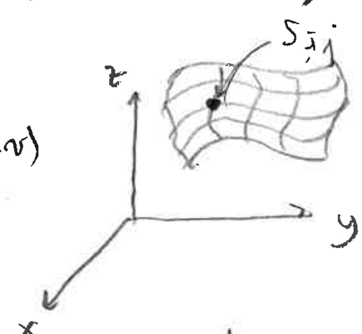
$$= 2\pi \, \frac{1}{2}\, \frac{2}{3}\left(1+r^2\right)^{3/2}\Big|_{r=0}^{r=1}$$

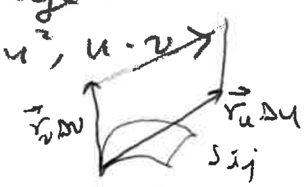
$$= \frac{2}{3}\pi\left(2\sqrt{2}-1\right)$$

Example: Find the equation of a plane tangent to the surface $\vec{r}(u,v) = \langle u+v,\ 3u^2,\ u-v\rangle$ at $(2,3,0)$.

Solution:




$dS = |\vec{r}_u \times \vec{r}_v|\, du\,dv$

$\vec{r}_u = \langle 1, 6u, 1\rangle$

$\vec{r}_v = \langle 1, 0, -1\rangle$

$$\left.\begin{aligned} u+v &= 2 \\ 3u^2 &= 3 \\ u-v &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} u &= 1 \\ v &= 1 \end{aligned}$$

$\vec{r}_u(1,1) = \langle 1, 6, 1\rangle$

$\vec{r}_v(1,1) = \langle 1, 0, -1\rangle$

$\vec{n} = \vec{r}_u \times \vec{r}_v = \langle -6, 2, -6\rangle$

The eq. of the plane:

$-6x+2y-6z$

$= -6(2)+2(3)-6(0)$

$= -12+6 = -6$

$3x - y + 3z = 3$

Quiz 12

Example: Find the area of the part of the plane

$\vec{r}(u,v) = \langle u+v,\ 3-v,\ 1+4u+5v \rangle$,

$0 \le u \le \sqrt{2}$, $-1 \le v \le 1$,

Solution: $A(S) = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$

$\vec{r}_u = \langle 1, 0, 4 \rangle$

$\vec{r}_v = \langle 1, -1, 5 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 4 \\ 1 & -1 & 5 \end{vmatrix} = \hat{i}(0+4) - \hat{j}(5-4) + \hat{k}(-1-0)$$

$= \langle 4, -1, -1 \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{16+1+1} = \sqrt{18}$

$A(s) = \int_{-1}^{1}\int_0^{\sqrt{2}} \sqrt{18}\, du\, dv = \sqrt{18}\,\sqrt{2}\,(1-(-1)) = 12$

Example: Find the area of the surface S given by

$\vec{r}(u,v) = \langle u^2, uv, \frac{1}{2}v^2 \rangle$, $0 \le u \le 1$, $0 \le v \le 2$.

Solution: $A(S) = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$

$\vec{r}_u = \langle 2u, v, 0 \rangle$

$\vec{r}_v = \langle 0, u, v \rangle$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = \hat{i}(v^2-0) - \hat{j}(2uv-0) + \hat{k}(2u^2-0)$$

$= \langle v^2, -2uv, 2u^2 \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2+2u^2)^2} = |v^2+2u^2|$

$= v^2 + 2u^2$

$A(s) = \int_0^2\int_0^1 (v^2+2u^2)\, du\, dv = \int_0^2 \left[uv^2 + \frac{2}{3}u^3\right]_0^1 dv$

$= \int_0^2 (v^2 + \frac{2}{3})\, dv = \frac{v^3}{3} + \frac{2}{3}v \Big|_0^2 = \frac{8}{3} + \frac{4}{3} = 4$

## §16.7 Surface Integral

Let $f(x,y,z)$. Then the integral of $f$ over a surface $S$ is defined as

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

Example: Evaluate $\iint_S (x+y+z)\,dS$ where $S$ is the parallelogram $x = u+v$, $y = u-v$, $z = 1+2u+v$, $0 \le u \le 2$, $0 \le v \le 1$.

Solution: $x+y+z = u+v+u-v+1+2u+v$

$= 4u+v+1$

$\vec{r}_u = \langle 1, 1, 2 \rangle$

$\vec{r}_v = \langle 1, -1, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 3, 1, -2 \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{9+1+4} = \sqrt{14}$

$$\iint_S (x+y+z)\,dS = \iint_D (4u+v+1)\sqrt{14}\,dA$$

$$= \int_0^1 \int_0^2 (4u+v+1)\sqrt{14}\,du\,dv$$

$$= \sqrt{14}\int_0^1 \left[2u^2+uv+u\right]_0^2 dv = \sqrt{14}\int_0^1 (10+2v)\,dv$$

$$= \sqrt{14}(10v+v^2)\Big|_0^1 = \sqrt{14}\,(11)$$