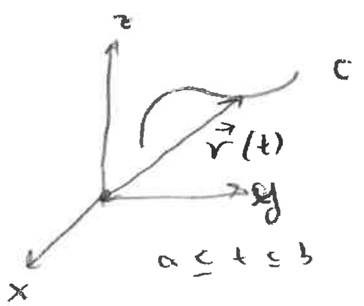
$$\nabla \times (\nabla f) = \vec{0}$$

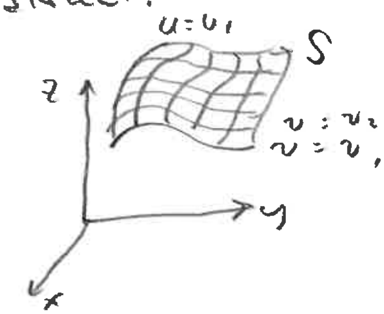
$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

Fri 11/21

§16.6 Surface Areas

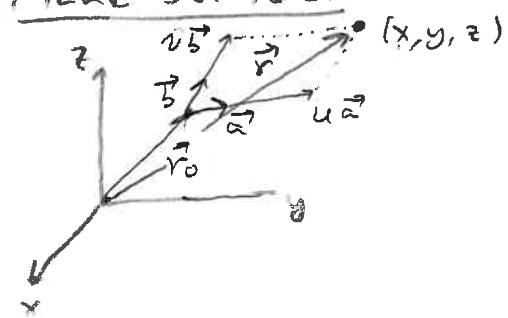
For a curve C, we parametrized it with $\vec{r}(t)$, because it is one-dimensional.





For a surface S, which is two dimensional, we need two parameters u and v. For a position vector $\vec{r}(u,v)$, if v is held constant, we trace a curve like $\vec{r}(t)$.

Plane Surface



$\vec{r}_0 = \vec{r}(u_0, v_0)$

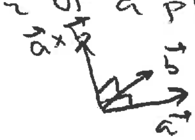
we need two vectors $\vec{a}$ and $\vec{b}$ in the plane that are not colinear.

$$\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$$

This is the parametrization of the plane.

To get the equation of a plane, we need a normal vector $\vec{n}$,

$$\vec{n} = \vec{a} \times \vec{b}$$



$\vec{n} \cdot \vec{a} = \vec{n} \cdot \vec{b} = 0$

The vector $\vec{r} - \vec{r}_0$ is in the plane

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

Example: Find the equation of a plane passing through $(0,-1,5)$ and containing $\langle 2,1,4\rangle$ and $\langle -3,2,5\rangle$.

Solution: $\vec{r}(u,v) = \langle 0,-1,5\rangle + u\langle 2,1,4\rangle + v\langle -3,2,5\rangle$

$$= \langle 0 + 2u - 3v,\ -1 + u + 2v,\ 5 + 4u + 5v\rangle$$

The parametric eqs are

$$x(u,v) = 2u - 3v$$
$$y(u,v) = u + 2v - 1$$
$$z(u,v) = 4u + 5v + 5$$

To find the equation we need $\vec{n}$,

$$\vec{n} = \langle 2,1,4\rangle \times \langle -3,2,5\rangle$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 4 \\ -3 & 2 & 5 \end{vmatrix} = \langle -3,-22,7\rangle$$

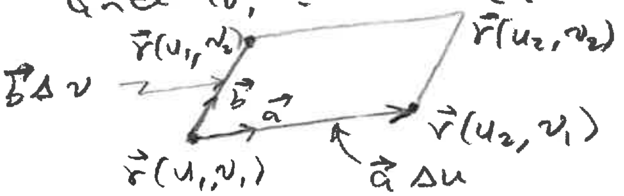
$$-3x - 22y + 7z = -3(0) - 22(-1) + 7(5) = 57$$

Let $u = v = 1$. Then

$$\vec{r}(1,1) = \langle -1, 2, 14\rangle.$$

$$-3(-1) - 22(2) + 7(14) = 3 - 44 + 98 = 57$$

We also would like to know the area of a region of the plane, which is a parallelogram. Let $u_1 \le u \le u_2$ and $v_1 \le v \le v_2$.



$$\vec{a} = \vec{r}(u_2,v_1) - \vec{r}(u_1,v_1)$$
$$= \vec{r}_0 + u_2\vec{a} + v_1\vec{b}$$
$$- (\vec{r}_0 + u_1\vec{a} + v_1\vec{b})$$
$$= (u_2 - u_1)\vec{a} = \vec{a}\,\Delta u$$

Area of parallelogram

$$\text{Area} = |\vec{a}\Delta u|\,|\vec{b}\Delta v| \sin\theta$$
$$= |\vec{a}\times\vec{b}|\,\Delta u \Delta v$$
$$= |\vec{n}|\,\Delta u \Delta v$$

Recall Green's Theorem,

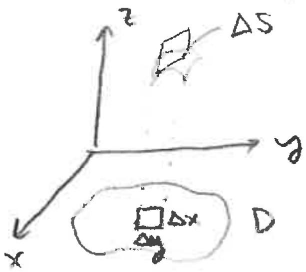
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (\nabla\times\vec{F})\cdot \underbrace{\hat{k}\, dA}$$

a differential surface vector $d\vec{S}$

$\hat{k}$ is normal to the surface

Graph of a Function Surface

Let $z = f(x,y)$. Here the parameters are x and y.



Approximate ΔS with a parallelogram. This is a linear approximation that we did in chapter 14.

$$z = f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y.$$

$$z = z_0 + \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

$$x = x_0 + x - x_0 = x_0 + \Delta x$$

$$y = y_0 + y - y_0 = y_0 + \Delta y$$

$$\vec{r}(x,y) = \left\langle x_0 + \Delta x,\ y_0 + \Delta y,\ z_0 + \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \right\rangle$$

$$= \langle x_0, y_0, z_0\rangle + \left\langle \Delta x, 0, \frac{\partial z}{\partial x}\Delta x\right\rangle + \left\langle 0, \Delta y, \frac{\partial z}{\partial y}\Delta y\right\rangle$$

$$= \vec{r_0} + \left\langle 1, 0, \frac{\partial z}{\partial x}\right\rangle \Delta x + \left\langle 0, 1, \frac{\partial z}{\partial y}\right\rangle \Delta y$$

This is a parametrization of the plane. So vectors $\left\langle 1, 0, \frac{\partial z}{\partial x}\right\rangle$ and $\left\langle 0, 1, \frac{\partial z}{\partial y}\right\rangle$ are in the plane.

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

$$\frac{\partial \vec{r}}{\partial x} = \vec{r}_x = \langle 1, 0, \frac{\partial f}{\partial x} \rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \vec{r}_y = \langle 0, 1, \frac{\partial f}{\partial y} \rangle$$

So $\vec{r}(x,y) = \vec{r}_0 + \vec{r}_x \Delta x + \vec{r}_y \Delta y$

So the area of ΔS is

$$A(\Delta S) = |\vec{r}_x \times \vec{r}_y| \Delta x \Delta y$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle$$

$$A(\Delta S) = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, \Delta x \Delta y$$

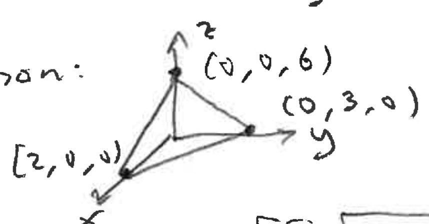
The total area of the surface S is

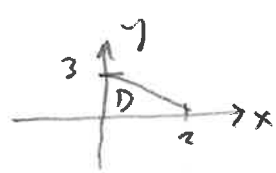
$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, \Delta x \Delta y$$

$$= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

Example: Find the area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Solution:





$z = 6 - 3x - 2y$

$$A(S) = \iint_D \sqrt{1 + (-3)^2 + (-2)^2} \, dA = \sqrt{14} \iint_D dA$$

$$= \sqrt{14}\, A(D) = \sqrt{14} \, \frac{1}{2}(2)(3) = 3\sqrt{14}$$