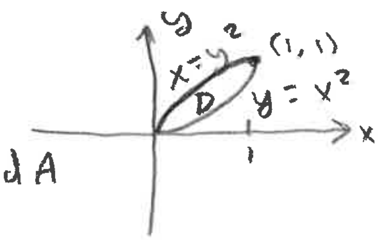
Example: Evaluate $\oint_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$ where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Solution: $\oint_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$



$$= \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}}) \right] dA$$

$$= \iint_D (2 - 1)\,dA = \iint_D dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx = \int_0^1 (x^{1/2} - x^2)\,dx$$

$$= \frac{2}{3} x^{3/2} - \frac{x^3}{3} \Big]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Wed 11/19

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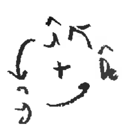
§16.5 Curl and Divergence

Let $\vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$.

The curl of $\vec{F}$ is

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial R}{\partial y} - \frac{\partial}{\partial z} Q \right) - \hat{\jmath}\left(\frac{\partial R}{\partial x} - \frac{\partial}{\partial z} P \right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$



$\hat{\jmath}\,\hat{k}\,\hat{\imath}$ $\quad$ $\hat{k}\,\hat{\imath}\,\hat{\jmath}$

$\hat{k}\,\hat{\jmath}\,\hat{\imath}$ $\quad$ $\hat{\imath}\,\hat{k}\,\hat{\jmath}$

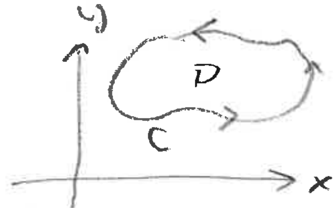
We saw yesterday that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ means $\vec{F}$ is conservative if $\vec{F}$ is defined on an open, simply connected region. This appears in Green's Theorem.

If $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, then Green's theorem is

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

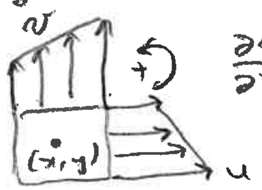
$(\nabla \times \vec{F})\cdot \hat{k} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ Then Green's theorem becomes

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (\text{curl } \vec{F})\cdot \hat{k}\, dA$$



Consider a velocity field in a fluid. Consider a small fluid element.

Let $\vec{V}(x,y) = u\,\hat{\imath} + v\,\hat{\jmath}$.



$$\frac{\partial v}{\partial x} = \frac{\partial Q}{\partial x} > 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial P}{\partial y} < 0 \quad \Rightarrow -\frac{\partial P}{\partial y} > 0$$

If $\nabla \times \vec{V} = \vec{0}$, the fluid is said to be irrotational.

Example: $\vec{V}(x,y,z) = u(x)\hat{\imath} + v(y)\hat{\jmath} + w(z)\hat{k}$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\cancelto{0}{\frac{\partial w}{\partial y}} - \cancelto{0}{\frac{\partial v}{\partial z}}\right)\hat{\imath} + \left(\cancelto{0}{\frac{\partial u}{\partial z}} - \cancelto{0}{\frac{\partial w}{\partial x}}\right)\hat{\jmath} + \left(\cancelto{0}{\frac{\partial v}{\partial x}} - \cancelto{0}{\frac{\partial u}{\partial y}}\right)\hat{k}$$

$$= 0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k} = \vec{0}$$

so the flow is irrotational.

Example: Find the curl of

$$\vec{F}(x,y,z) = xy^2z^2\hat{i} + x^2yz^2\hat{j} + x^2y^2z\hat{k}.$$

Solution: $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial(x^2y^2z)}{\partial y} - \frac{\partial(x^2yz^2)}{\partial z}\right) - \hat{j}\left(\frac{\partial}{\partial x}(x^2y^2z) - \frac{\partial}{\partial z}(xy^2z^2)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(x^2yz^2) - \frac{\partial}{\partial y}(xy^2z^2)\right)$$

$$= \hat{i}\left(2x^2yz - 2x^2yz\right) - \hat{j}\left(2xy^2z - 2xy^2z\right)$$

$$+ \hat{k}\left(2xyz^2 - 2xyz^2\right)$$

$$= \vec{0}$$

Thus $\vec{F}$ is conservative since is defined on $\mathbb{R}^3$ (an open simply connected region). There exist f such that $\nabla f = \vec{F}$.

$\frac{\partial f}{\partial x} = xy^2z^2 \Rightarrow f(x,y,z) = \frac{x^2}{2}y^2z^2 + g(y,z)$

$\frac{\partial f}{\partial y} = x^2yz^2 + \frac{\partial g}{\partial y} = x^2yz^2 \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow$

$g(y,z) = h(z)$. Then $f(x,y,z) = \frac{x^2}{2}y^2z^2 + h(z)$.

$\frac{\partial f}{\partial z} = x^2y^2z + h'(z) = x^2y^2z \Rightarrow h'(z) = 0 \Rightarrow$

$h(z) = K$ a constant.

$f(x,y,z) = \frac{x^2y^2z^2}{2} + K.$

Divergence

Using the same $\vec{F}(x,y,z)$, the divergence is

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Example: Find divergence of $\vec{F}$ in previous example.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xy^2z^2) + \frac{\partial}{\partial y}(x^2yz^2) + \frac{\partial}{\partial z}(x^2y^2z)$$

$$= y^2z^2 + x^2z^2 + x^2y^2$$

Since $\vec{F}$ is conservative, $\nabla f = \vec{F}$. Then

$$\nabla \cdot \vec{F} = \nabla \cdot (\nabla f) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

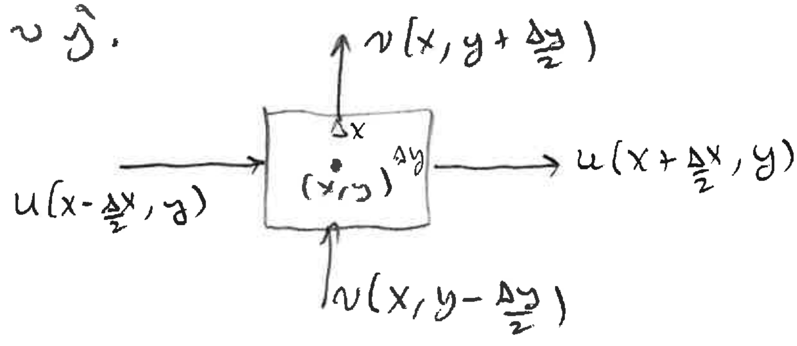
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}_{\text{Laplacian operator}} f$$

$$= \nabla^2 f$$

the Laplacian of f

Why is it called divergence. Consider the fluid field.

$\vec{V} = u\hat{i} + v\hat{j}$.



$$\lim_{\Delta x, \Delta y \to 0} \frac{[u(x+\frac{\Delta x}{2}, y) - u(x - \frac{\Delta x}{2}, y)]\Delta y}{\Delta x \Delta y} + \frac{[v(x, y+\frac{\Delta y}{2}) - v(x, y - \frac{\Delta y}{2})]\Delta x}{\Delta x \Delta y}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \vec{V}$$

This is a measure of how much the shape of the fluid diverges from its original shape.

If $\nabla \cdot \vec{V} = 0$, the fluid is incompressible, as a liquid.

$\nabla \times (\nabla f) = \vec{0}$

$\nabla \cdot (\nabla \times \vec{F}) = 0$