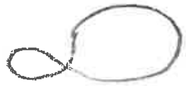
Tues 11/18

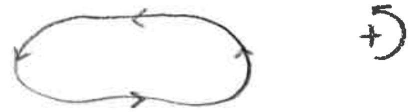
§16.4 Green's theorem



closed curve

$\vec{r}(a) = \vec{r}(b)$

$a \le t \le b$



simple closed curve

positively oriented

we saw for $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ a conservative vector field that there exists $f(x,y)$ such that $\nabla f = \vec{F}$. And

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

so if C is closed, $f(\vec{r}(b)) = f(\vec{r}(a))$, we have

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

we write the integral over a closed curve as

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}.$$

So for a conservative vector field $\oint_C \vec{F} \cdot d\vec{r} = 0$.

We also saw that for a conservative vector field,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Suppose $\vec{F}$ is not conservative, then

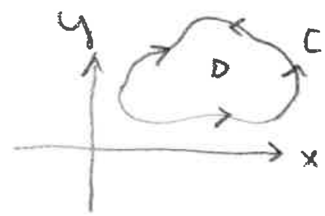
$$\oint_C \vec{F} \cdot d\vec{r} \neq 0 \quad \text{and} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \neq 0.$$

So we might suppose there is a relation between these two. Consider the units for a force field. $\vec{F} \cdot d\vec{r}$ has units $N \cdot m = J$ and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ has units $\frac{N}{m}$

We need to multiply $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ by area dA.

Green's theorem

Let C be a positively oriented, piecewise smooth, simple closed curve enclosing a region D. In other words, C is the boundary of D, some people write it as ∂D.

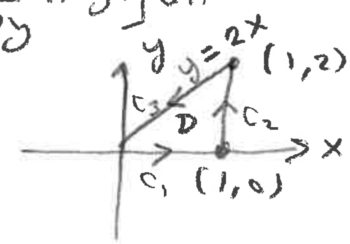


Let $F(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$ have continuous partial derivatives. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Example: Evaluate $\oint_C xy\,dx + x^2y^3\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.

Solution: $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA$



$= \iint_D (2xy^3 - x)\,dA$

$= \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$= \int_0^1 \left[\frac{xy^4}{2} - xy\right]_0^{2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx$

$= \frac{8}{6}x^6 - \frac{2}{3}x^3 \Big|_0^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

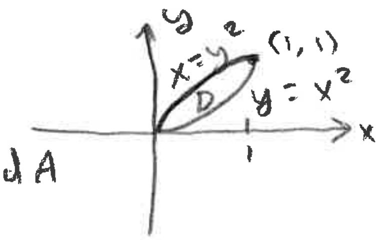
$\oint_C xy\,dx + x^2y^3\,dy = \int_{C_1} 0\,dx + 0\,dy + \int_{C_2} y\,dx + y^3\,dy$

$+ \int_{C_3} 2x^2\,dx + 8x^5\,dy = \int_1^1 y\,\overset{0}{\cancel{dx}} + \int_0^2 y^3\,dy$

$+ \int_1^0 2x^2\,dx + 8x^5\,2\,dx$

$= \frac{y^4}{4}\Big|_0^2 + \left[\frac{2}{3}x^3 + \frac{8}{3}x^6\right]_1^0 = 4 - \frac{2}{3} - \frac{8}{3} = \frac{2}{3}$

Example: Evaluate $\oint_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$ where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.



Solution: $\oint_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$

$$= \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}})\right] dA$$

$$= \iint_D (2 - 1)\,dA = \iint_D dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx = \int_0^1 (x^{1/2} - x^2)\,dx$$

$$= \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$