Example: Find $\int_C y\,dx + z\,dy + x\,dz$ where

$C: x = \sqrt{t},\ y = t,\ z = t^2,\ 1 \le t \le 4$

Solution: $\int_1^4 t \frac{1}{2\sqrt{t}}dt + t^2 dt + \sqrt{t}\, 2t\,dt$

$= \int_1^4 \left(\frac{1}{2}t^{1/2} + t^2 + 2t^{3/2}\right)dt$

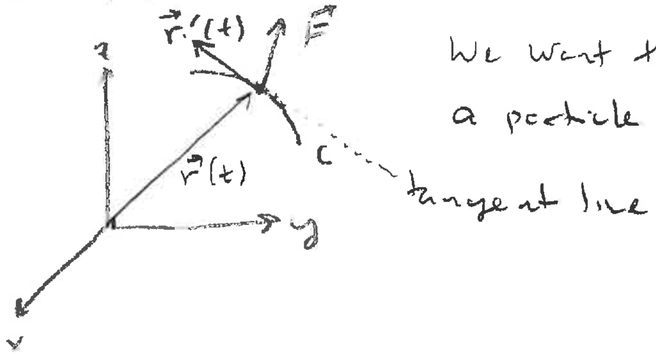
$= \left[\frac{1}{3}t^{3/2} + \frac{t^3}{3} + \frac{4}{5}t^{5/2}\right]_1^4$

$= \frac{8}{3} + \frac{64}{3} + \frac{128}{5} - \left(\frac{1}{3} + \frac{1}{3} + \frac{4}{5}\right)$

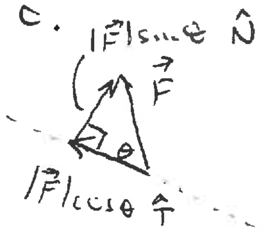
$= \frac{70}{3} + \frac{124}{5} = \frac{722}{15}$

Fri 11/14

Let $\vec{F}(\vec{r})$ be a force field and C a smooth curve.



We want the work needed to move a particle along C.



The work need to move the particle a distance ds is

$$dW = |\vec{F}|\cos\theta\, ds = |\vec{F}|\frac{|\vec{r}'(t)|}{|\vec{r}'(t)|}\cos\theta\, ds$$

$$= \vec{F}\cdot\frac{\vec{r}'(t)}{|\vec{r}'(t)|}ds = \vec{F}\cdot\frac{\vec{r}'(t)}{ds/dt}ds = \vec{F}\cdot\vec{r}'(t)\,dt$$

The total work is then

$$W = \int_C \vec{F}\cdot\vec{r}'(t)\,dt = \int_C \vec{F}\cdot\frac{d\vec{r}}{dt}dt = \int_C \vec{F}\cdot d\vec{r}$$

If $\vec{F}(x,y,z) = P(x,y,z)\,\hat{i} + Q(x,y,z)\,\hat{j} + R(x,y,z)\,\hat{k}$,

then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

Example: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \sin x\,\hat{i} + \cos y\,\hat{j} + xz\,\hat{k}$

and $\vec{r}(t) = t^3\,\hat{i} - t^2\,\hat{j} + t\,\hat{k}$, $0 \le t \le 1$.

Solution: $x(t) = t^3$, $y(t) = -t^2$, $z(t) = t$

$$\vec{F}(t) = \langle \sin t^3, \cos(-t^2), t^4 \rangle$$

$$= \langle \sin t^3, \cos t^2, t^4 \rangle$$

$$d\vec{r} = \langle 3t^2 dt, -2t\,dt, dt \rangle = \langle 3t^2, -2t, 1 \rangle\, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 \sin t^3 - 2t\cos t^2 + t^4)\, dt$$

$$= -\cos t^3 - \sin t^2 + \frac{t^5}{5} \Big|_0^1$$

$$= -\cos 1 - \sin 1 + \frac{1}{5} - (-1 + 0 + 0)$$

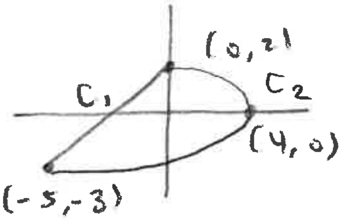
$$= \frac{6}{5} - \cos 1 - \sin 1$$

Example: Let $\vec{F}(x,y) = y^2\,\hat{i} + x\,\hat{j}$ and let

(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$

(b) $C = C_2$ is the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Calculate $\int_C \vec{F}\cdot d\vec{r}$ for both paths and compare



C_1: $y = x + 2$

Solution: (a) $\int_{C_1} y^2\,dx + x\,dy$

$$= \int_{-5}^{0} (x+2)^2\,dx + \int_{-3}^{2} (y-2)\,dy$$

$$= \frac{(x+2)^3}{3}\Big|_{-5}^{0} + \frac{(y-2)^2}{2}\Big|_{-3}^{2}$$

$$= \left(\frac{8}{3} + 9\right) + 0 - \frac{25}{2}$$

$$= \frac{35}{3} - \frac{25}{2} = \frac{70}{6} - \frac{75}{6} = -\frac{5}{6}$$

(b) $\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2(-2y\,dy) + \int_{-3}^{2} (4-y^2)\,dy$

$$= -2\int_{-3}^{2} y^3\,dy + \int_{-3}^{2} (4-y^2)\,dy$$

$$= -\frac{y^4}{2} + 4y - \frac{y^3}{3}\Big|_{-3}^{2}$$

$$= \left(-8 + \frac{81}{2}\right) + \left(8 - \frac{8}{3} + 12 - 9\right)$$

$$= \frac{65}{2} + \frac{25}{3}$$

$$= \frac{195}{6} + \frac{50}{6} = \frac{245}{6} \neq -\frac{5}{6}$$

This integral $\int_C \vec{F}\cdot d\vec{r}$ depends on path.

Example: Rework the problem with $\vec{F} = y^2\hat{i} + 2xy\,\hat{j}$

Solution: (a) $\int_{C_1} y^2 dx + 2xy\,dy = \frac{35}{3} + \int_{C_1} 2xy\,dy$

$$= \frac{35}{3} + \int_{-3}^{2} 2(y-2)y\,dy$$

$$= \frac{35}{3} + 2\int_{-3}^{2}(y^2-2y)\,dy$$

$$= \frac{35}{3} + 2\left(\frac{y^3}{3} - y^2\right)\Big|_{-3}^{2}$$

$$= \frac{35}{3} + 2\left(\frac{8}{3} - 4 + 9 + 9\right)$$

$$= \frac{35}{3} + \frac{100}{3} = \frac{135}{3} = 45$$

(b) $\int_{C_2} y^2 dx + 2xy\,dy = \frac{65}{2} + \int_{C_2} 2xy\,dy$

$$= \frac{65}{2} + 2\int_{-3}^{2} (4-y^2)y\,dy$$

$$= \frac{65}{2} + 2\int_{-3}^{2} (4y - y^3)\,dy$$

$$= \frac{65}{2} + 2\left(2y^2 - \frac{y^4}{4}\right)\Big|_{-3}^{2}$$

$$= \frac{65}{2} + 2\left(8 - 4 - 18 + \frac{81}{4}\right)$$

$$= \frac{65}{2} + 2\left(\frac{25}{4}\right) = \frac{65}{2} + \frac{25}{2} = \frac{90}{2} = 45$$

This integral is path independent. Consider the difference.

$$\vec{F}_1 = y^2\hat{i} + x\hat{j}$$

$$\vec{F}_2 = y^2\hat{i} + 2xy\,\hat{j}$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \Rightarrow f(x,y) =$$

$$\frac{\partial f}{\partial x} = y^2 \Rightarrow f(x,y) = xy^2 + g(y)$$

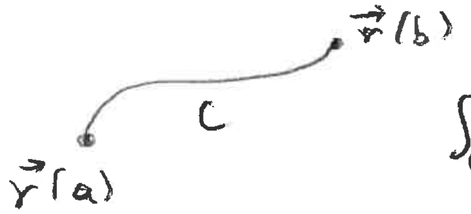
$$\frac{\partial f}{\partial y} = 2xy + g'(y) = 2xy \Rightarrow g'(y) = 0 \Rightarrow g(y) = 1$$

So $f(x,y) = xy^2$

In other words $\vec{F}_2$ is conservative because

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = y^2\hat{i} + 2xy\,\hat{j} = \vec{F}_2(x,y)$$

No such f exists for $\vec{F}_1$. So $\vec{F}_1$ is not a conservative vector field.



If $\nabla f = \vec{F}$, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

Compare this to Fundamental theorem of Calculus

$$\int_a^b F\,dx = \int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$$