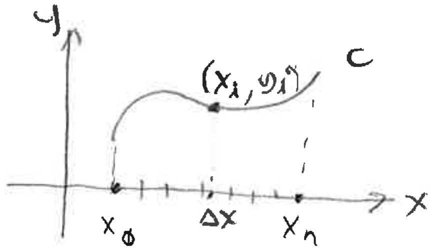
Wed 11/12

The line integral

$$\int_C f(x,y)\,ds$$

integrates along a curve ~~c~~ with respect to arc length.
We can also integrate along C with respect to x or y.



$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i)\Delta x = \int_C f(x,y)\,dx$$

$$= \int_a^b f(x(t),y(t))\,\frac{dx}{dt}\,dt$$

$$\leq \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Similarly, we can $\int_C f(x,y)\,dy$. Often will evaluate these two together.

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

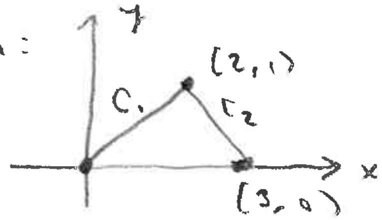
For $\vec{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$ and $\vec{r} = x\hat{i} + y\hat{j} \Rightarrow$
$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$, then

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C \vec{F}\cdot d\vec{r}$$

If $\vec{F}$ is a force field, then this integral calculates the work done moving a particle along a curve C.

Example: Find $\int_C (x+2y)\,dx + x^2\,dy$ where C consists of line segments from $(0,0)$ to $(2,1)$ and from $(2,1)$ to $(3,0)$.

Solution:



$C_1: y = \frac{1}{2}x$

$C_2: y = -x + 3$

$$\int_C (x+2y)\,dx + x^2\,dy = \int_{C_1} (x+2y)\,dx + x^2\,dy + \int_{C_2} (x+2y)\,dx + x^2\,dy$$

$$= \int_0^2 [x + 2(\tfrac{1}{2}x)]\,dx + x^2 \tfrac{1}{2}\,dx + \int_2^3 [x + 2(3-x)\,dx + x^2(-dx)$$

$$= \int_0^2 \left(\frac{x^2}{2} + 2x\right)dx + \int_2^3 (-x^2 - x + 6)\,dx$$

$$= \left[\frac{x^3}{6} + x^2\right]_0^2 + \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x\right]_2^3$$

$$= \frac{4}{3} + 4 + \left(-9 - \frac{9}{2} + 18 + \frac{8}{3} + 2 - 12\right)$$

$$= 7 - \frac{9}{2} = \frac{14}{2} - \frac{9}{2} = \frac{5}{2}$$

Line Integrals in Space

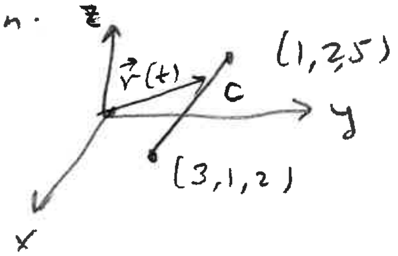
$$\int_C f(x,y,z)\,ds$$

$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Example: $\int_C y^2 z\, ds$, where C is the line segment from $(3,1,2)$ to $(1,2,5)$.

Solution.



$\vec{r}(0) = \langle 3,1,2 \rangle$

$\vec{r}(1) = \langle 1,2,5 \rangle$

$$\vec{r}(t) = \langle 3,1,2 \rangle (1-t) + \langle 1,2,5 \rangle t$$

$$= \langle 3,1,2 \rangle + t \langle 1-3, 2-1, 5-2 \rangle$$

$$= \langle 3-2t, 1+t, 2+3t \rangle$$

$x(t) = 3-2t$

$y(t) = 1+t$

$z(t) = 2+3t$

$$ds = \sqrt{(-2)^2 + 1^2 + 3^2}\, dt = \sqrt{14}\, dt$$

$$y^2 z = (1+t)^2(2+3t) = (1+2t+t^2)(2+3t)$$

$$= 2 + 4t + 2t^2 + 3t + 6t^2 + 3t^3$$

$$= 3t^3 + 8t^2 + 7t + 2$$

$$\int_C y^2 z\, ds = \int_0^1 (3t^3 + 8t^2 + 7t + 2)\sqrt{14}\, dt$$

$$= \sqrt{14} \left[\frac{3}{4}t^4 + \frac{8}{3}t^3 + \frac{7}{2}t^2 + 2t \right]_0^1$$

$$= \sqrt{14} \left(\frac{3}{4} + \frac{8}{3} + \frac{7}{2} + 2 \right)$$

$$= \frac{9 + 32 + 42 + 24}{12}\sqrt{14} = \frac{107}{12}\sqrt{14}$$

Example: Find $\int_C y\,dx + z\,dy + x\,dz$ where

$C: x = \sqrt{t},\ y = t,\ z = t^2,\ 1 \le t \le 4$

Solution: $\int_1^4 t \frac{1}{2\sqrt{t}}\,dt + t^2\,dt + \sqrt{t}\,2t\,dt$

$= \int_1^4 \left(\frac{1}{2} t^{1/2} + t^2 + 2t^{3/2}\right) dt$

$= \left[\frac{1}{3} t^{3/2} + \frac{t^3}{3} + \frac{4}{5} t^{5/2}\right]_1^4$

$= \frac{8}{3} + \frac{64}{3} + \frac{128}{5} - \left(\frac{1}{3} + \frac{1}{3} + \frac{4}{5}\right)$

$= \frac{70}{3} + \frac{124}{5} = \frac{722}{15}$