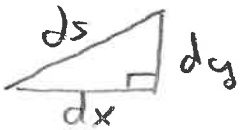
the area is approximately,

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i, y_i)\Delta s$$

which is a Riemann sum. the exact area is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta s = \int_C f(x,y)\,ds$$

This is the definition of a line integral.
A differential section ds of the curve C is a straight line.



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

Tues 11/11

The line integral of $f(x,y)$ with respect to arc length along a curve C is

$$\int_C f(x,y)\,ds.$$

Recall from ch 13 that the speed of a particle traveling along C is

$$\frac{ds}{dt} = |\vec{r}\,'(t)| \quad , \text{ where } \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$
$$\vec{r}\,'(t) = x'(t)\hat{i} + y'(t)\hat{j}$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Then the line integral becomes

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

where $a \le t \le b$.

Example: Evaluate $\int_C y\, ds$, $C: x = t^2, y = 2t, 0 \le t \le 3$.

Solution:
$$\int_C y\, ds = \int_0^3 2t \sqrt{4t^2+4}\, dt$$
$$= 2\int_0^3 2t\,(t^2+1)^{1/2}\, dt$$
$$= 2\int_0^3 (t^2+1)^{1/2}\, d(t^2+1)$$
$$= 2\left.\frac{(t^2+1)^{3/2}}{3/2}\right|_0^3 = \frac{4}{3}\left(10^{3/2}-1\right)$$

We can also convert ds to dx. Let $y = g(x)$.

$$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+(g'(x))^2}\, dx$$

Then the line integral is

$$\int_C f(x,y)\, ds = \int_{x_1}^{x_2} f(x, g(x))\sqrt{1+(g'(x))^2}\, dx$$

Rework the previous example with this substitution.

$$y = 2t = 2\sqrt{x}, \quad 0 \le x \le 9$$
$$\int_C y\, ds = \int_0^9 2\sqrt{x}\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\, dx = 2\int_0^9 \sqrt{x+1}\, dx$$
$$= 2\left.\frac{(x+1)^{3/2}}{3/2}\right|_0^9 = \frac{4}{3}\left(10^{3/2}-1\right)$$

Similarly, we can write for $x = g(y)$,

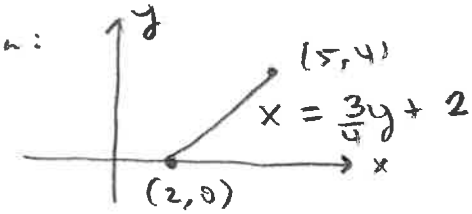
$$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{\left(\frac{dx}{dy}\right)^2+1}\, dy.$$

Then the line integral becomes

$$\int_{y_1}^{y_2} F(g(y), y)\sqrt{[g'(y)]^2+1}\, dy.$$

Example: Evaluate $\int_C x e^y \, ds$, for C the line segment from $(2,0)$ to $(5,4)$.

Solution:



$$\int_C x e^y \, ds = \int_0^4 \left(\tfrac{3}{4}y+2\right) e^y \sqrt{\left(\tfrac{3}{4}\right)^2 + 1} \, dy$$

$$= \frac{5}{4} \int_0^4 \left(\tfrac{3}{4}y+2\right) de^y$$

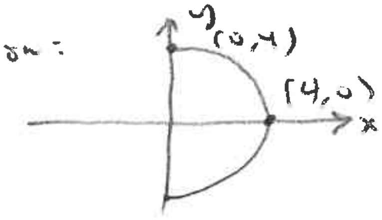
$$= \frac{5}{4}\left(\tfrac{3}{4}y+2\right)e^y \Big|_0^4 - \frac{5}{4}\int_0^4 e^y \, d\left(\tfrac{3}{4}y+2\right)$$

$$= \frac{25}{4}e^4 - \frac{5}{2} - \frac{15}{16}\int_0^4 e^y \, dy$$

$$= \frac{25}{4}e^4 - \frac{5}{2} - \frac{15}{16}\left(e^4 - 1\right) = \frac{85}{16}e^4 - \frac{25}{16}$$

Example: Evaluate $\int_C x y^4 \, ds$ where C is the right half of the circle $x^2 + y^2 = 16$.

Solution:



$x = 4\cos t$

$y = 4\sin t$

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-4\sin t)^2 + (4\cos t)^2}$$

$$= \sqrt{16\sin^2 t + 16\cos^2 t}$$

$$= 4\sqrt{1} = 4$$

$ds = 4\,dt$

$$\int_C x y^4 \, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4 \, 4\,dt$$

$$= 4^6 \int_{-\frac{\pi}{2}}^{\pi/2} \cos t \left(\sin^4 t\right) dt = 4^6 \int_{-\pi/2}^{\pi/2} \sin^4 t \, d\sin t$$

$$= 4^6 \, \frac{\sin^5 t}{5} \Big|_{-\pi/2}^{\pi/2} = \frac{4^6}{5}\left(1 - (-1)\right) = \frac{2}{5}(4)^6$$