§16.1 Vector Fields

In Chapter 12-13, we looked at vector-valued functions

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$

In chapters 14-15, we looked at multidimensional functions

$$f(x,y,z).$$

In chapter 16, we will look at vector-valued, multidimensional functions

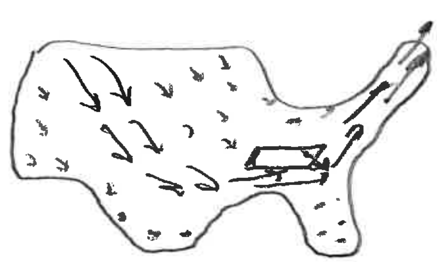
$$\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}.$$

This type of function is called a vector field.

The most common vector field is a velocity field.

We graph vector fields by superimposing the output vectors on the input points.

Consider a region D that is the USA. At any point (x,y) in D we graph the velocity of the wind $\vec{V}(x,y)$.

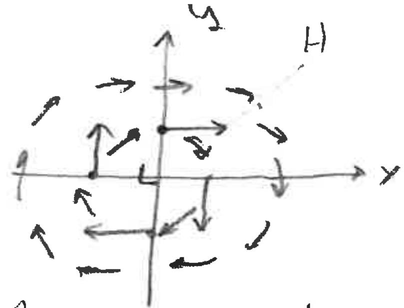


This is a graph of a vector field.

Example: $\vec{F}(x,y) = \dfrac{y\hat{\imath} - x\hat{\jmath}}{\sqrt{x^2+y^2}}$. Sketch vector field.

Solution: $|\vec{F}| = \sqrt{\dfrac{y^2}{x^2+y^2} + \dfrac{x^2}{x^2+y^2}} = 1$ a unit vector

(x,y)	$\langle y,-x \rangle$
$(1,0)$	$\langle 0,-1 \rangle$
$(0,1)$	$\langle 1,0 \rangle$
$(-1,0)$	$\langle 0,1 \rangle$
$(0,-1)$	$\langle -1,0 \rangle$



This is a vortex such as hurricane, tornado, whirlpool, or toilet.

This is because of Coriolis acceleration.

Gradient Vector Field

$\vec{F}$ is a gradient vector field if there exists f such that $\vec{F} = \nabla f = \dfrac{\partial f}{\partial x}\hat{\imath} + \dfrac{\partial f}{\partial y}\hat{\jmath} + \dfrac{\partial f}{\partial z}\hat{k}$

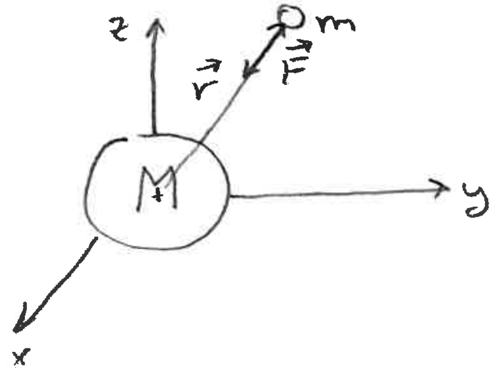
This is a conserved vector field. And the function f is called the potential of $\vec{F}$.

An example is an electric field $\vec{E} = \nabla V$ where V is the voltage.

Example: Let $f(x,y,z) = \sqrt{x^2+y^2+z^2} = |\vec{r}|$. Find the gradient vector field.

Solution: $\vec{F} = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}}\hat{\imath} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\hat{\jmath} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$

A gravitational field is a vector field.



$$|\vec{F}| = \frac{mMG}{|\vec{r}|^2}$$

$$\vec{F} = |\vec{F}| \frac{(-\vec{r})}{|\vec{r}|}$$

unit vector in direction of $\vec{F}$

$$\vec{F} = \frac{-mMG\vec{r}}{|\vec{r}|^3}$$ Is this a conserved vector field?

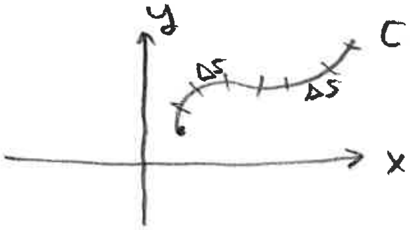
Can we find f such that $\dfrac{-mMG\vec{r}}{|\vec{r}|^3} = \nabla f$?

$$\vec{F} = \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}\hat{i} - \frac{mMGy}{(x^2+y^2+z^2)^{3/2}}\hat{j} - \frac{mMGz}{(x^2+y^2+z^2)^{3/2}}\hat{k}$$

$$= \nabla f \quad \text{where } f = \frac{mMG}{(x^2+y^2+z^2)} = \frac{mMG}{|\vec{r}|}$$

§16.2 Line Integrals

Suppose we want to measure the area of a fence or a curtain that is erected along a curve C with height $f(x,y)$ where (x,y) is on C.



We will walk the curve from time t_0 to t_n. We measure the length of the curve along the way. Doing so, gives an arc length function $s(t)$.

So $s(t_0) = 0$, and $s(t_n) = L$ the length of the curve. Let's partition the curve C into n sections of equal length Δs. At each section i, we measure the height $f(x_i, y_i)$ where (x_i, y_i) is on C in the i^{th} section.

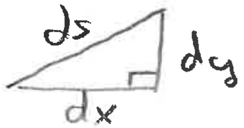
the area is approximately,

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i, y_i)\Delta s$$

which is a Riemann sum. the exact area is

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta s = \int_C f(x,y)\, ds$$

This is the definition of a line integral.
A differential section ds of the curve C is
a straight line.



$$ds = \sqrt{(dx)^2 + (dy)^2}$$