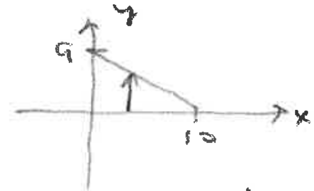
Tues 11/4

(14) $I_x = \iint_D \rho y^2\,dA$




$$= \int_0^{10}\int_0^{\frac{9}{10}(10-x)} 9y^2\,dy\,dx$$

$$\frac{x}{10}+\frac{y}{9}=1 \Rightarrow y = 9\left(1-\frac{x}{10}\right) = \frac{9}{10}(10-x)$$

$$= 9\int_0^{10} \frac{y^3}{3}\Big|_0^{\frac{9}{10}(10-x)} dx$$

$$= 3\left(\frac{9}{10}\right)^3 \int_0^{10} (10-x)^3\,dx = 3\left(\frac{9}{10}\right)^3 \frac{(10-x)^4}{-4}\Big|_0^{10}$$

$$= \frac{3}{4}\left(\frac{9}{10}\right)^3 10^4 = \frac{3}{4}(9)(81)(10) = \frac{3(9)(810)}{4} = \frac{27(405)}{2}$$

$$= \frac{10935}{2}$$

Quiz 10 Tuesday 11/11

§15.9 Change of variables

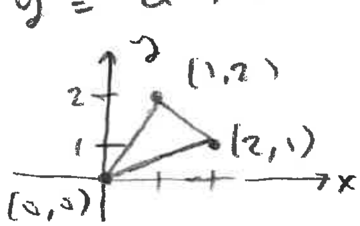
$$\iint_D f(x,y)\,dA = \int_{v_1}^{v_2}\int_{u_1}^{u_2} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Examples: Evaluate $\iint_D (x-3y)\,dA$ where $D$ is the triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$, using the transformation

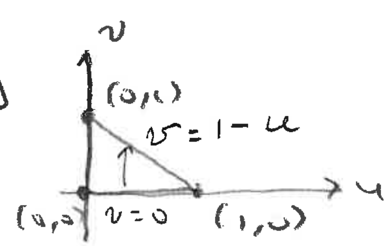
$x = 2u+v$, $y = u+2v$.

Solutions: $\left.\begin{array}{l} x = 2u+v \\ 2y = 2u+4v \end{array}\right\} \Rightarrow x-2y = -3v \Rightarrow v = -\frac{1}{3}x + \frac{2}{3}y$

$\left.\begin{array}{l} 2x = 4u+2v \\ y = u+2v \end{array}\right\} \Rightarrow 2x-y = 3u \Rightarrow u = \frac{2}{3}x - \frac{1}{3}y$




$$\xrightarrow[v=-\frac{1}{3}x+\frac{2}{3}y]{u=\frac{2}{3}x-\frac{1}{3}y}$$




$u(2,1) = \frac{4}{3} - \frac{1}{3} = 1$ $\quad u(1,2) = 0$

$v(2,1) = -\frac{2}{3} + \frac{2}{3} = 0$ $\quad v(1,2) = -\frac{1}{3} + \frac{4}{3} = 1$

$$x - 3y = 2u+v - 3(u+2v) = -u-5v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3$$

$$\int_0^1\int_0^{1-u} -(u+5v)\,|3|\,dv\,du = -3\int_0^1 \left[uv + \frac{5v^2}{2}\right]_0^{1-u} du$$

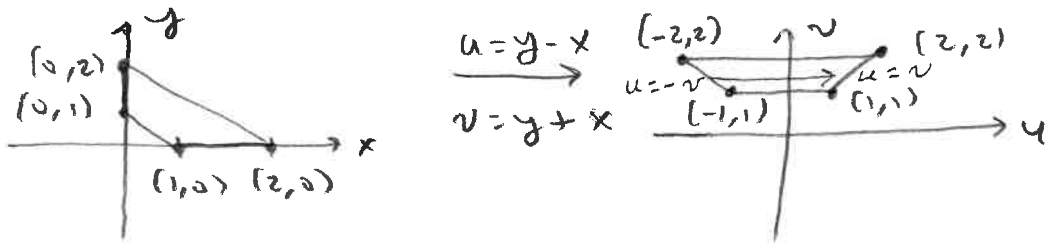
$$= -3\int_0^1 \left[u - u^2 + \frac{5}{2}(1-u)^2\right] du$$

$$= -3\left[\frac{u^2}{2} - \frac{u^3}{3} - \frac{5}{2}\frac{(1-u)^3}{3}\right]_0^1$$

$$= -3\left[\frac{1}{2} - \frac{1}{3} - 0 - 0 + 0 + \frac{5}{6}\right] = -3$$

Example: Evaluate $\iint_D \cos\left(\frac{y-x}{y+x}\right) dA$ where $D$ is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2$, and $(0,1)$.

Solution: Let $u = y-x$ and $v = y+x$. then $x = \frac{v-u}{2}$ and $y = \frac{v+u}{2}$.




$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\int_1^2\int_{-v}^{v} \left(\cos\frac{u}{v}\right)\left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\int_1^2 v\sin\frac{u}{v}\Big|_{-v}^{v} dv$$

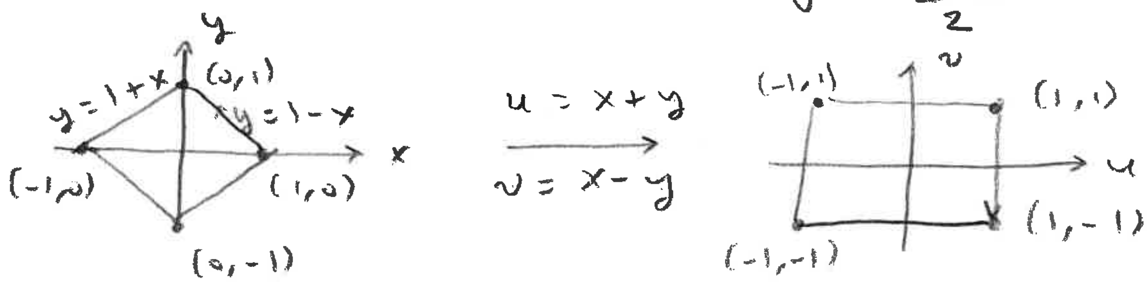
$$= \frac{1}{2}\int_1^2 [v\sin 1 - v\sin(-1)]\,dv$$

$$= \frac{1}{2}\int_1^2 (v\sin 1 + v\sin 1)\,dv$$

$$= \int_1^2 v\sin 1\,dv = \frac{v^2}{2}\sin 1\Big|_1^2 = \frac{3}{2}\sin 1$$

Example: Evaluate $\iint_D e^{x+y}\,dA$, where $D = \{(x,y) \mid |x|+|y| \le 1\}$.

Solution: Let $u = x+y$, $v = x-y$. Then $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$.




$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\int_{-1}^{1}\int_{-1}^{1} e^u \left|-\tfrac{1}{2}\right| du\,dv = \frac{1}{2}\int_{-1}^{1} e^u \Big|_{-1}^{1} dv$$

$$= \frac{1}{2}\int_{-1}^{1}\left(e - \frac{1}{e}\right)dv = \frac{1}{2}\left(e - \frac{1}{e}\right)v\Big|_{-1}^{1} = e - \frac{1}{e}$$