Example: Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$

Solution: $\sqrt{x^2+y^2} = r \le z \le 2$

$-\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \Rightarrow x^2+y^2 = r^2 = 4 \Rightarrow r = 2$

$-2 \le y \le 2 \Rightarrow r = 2 \qquad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{2}\int_r^{2} (r\cos\theta)\, z r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{2} r^2\cos\theta\left(\frac{4-r^2}{2}\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \cos\theta\left(2r^2 - \frac{r^4}{2}\right) dr\, d\theta$$

$$= \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} \int_0^{2} \left(2r^2 - \frac{r^4}{2}\right) dr = 0$$

Wed 10/29

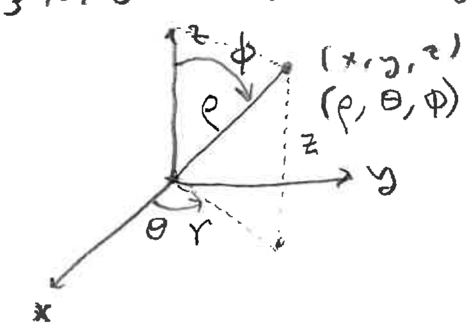
Test III Wednesday 11/5 covers §15.1 - 15.4, 15.6 - 15.8.

Review sheet tomorrow will go over Monday 11/3.

Tuesday study day. 11/4

§15.9 is not on exam, Quiz 10 on Tuesday 11/11.

§15.8 Triple Integrals in Spherical Coordinates



We need another angle ϕ to relate ρ to r and subsequently ρ to x and y. ϕ is called the polar angle.

$$0 \le \phi \le \pi$$

The angle θ is the azimuthal angle

$$0 \le \theta \le 2\pi$$

We see that

$\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ is the equation of a sphere.

$r = \rho \sin\phi$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

Example: Convert $(\rho,\theta,\phi) = (6, \pi/3, \pi/6)$ to (x,y,z).

Solution: $x = 6\sin\frac{\pi}{6}\cos\frac{\pi}{3} = 6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{2}$

$y = 6\sin\frac{\pi}{6}\sin\frac{\pi}{3} = 6\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$z = 6\cos\frac{\pi}{6} = 6\frac{\sqrt{3}}{2} = 3\sqrt{3}$

$$\left.\begin{array}{l} x^2 = \rho^2\sin^2\phi\cos^2\theta \\ y^2 = \rho^2\sin^2\phi\sin^2\theta \end{array}\right\} x^2+y^2 = \rho^2\sin^2\phi(\overbrace{\cos^2\theta+\sin^2\theta}^{1}) \Rightarrow$$

$z^2 = \rho^2\cos^2\phi$

$$x^2+y^2+z^2 = \rho^2(\overbrace{\sin^2\phi+\cos^2\phi}^{1}) = \rho^2$$

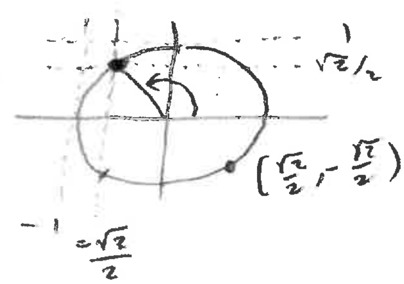
$$\boxed{\begin{array}{l} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\frac{y}{x} \\ \phi = \cos^{-1}\frac{z}{\rho} \end{array}}$$

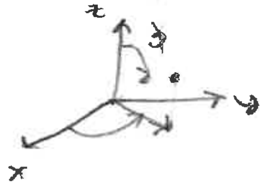
Example: Convert $(x,y,z) = (-1, 1, -\sqrt{2})$ to (ρ,θ,ϕ).

Solution: $\rho = \sqrt{1+1+2} = 2$

$\theta = \tan^{-1}\frac{1}{-1} = \tan^{-1}\frac{\sqrt{2}/2}{-\sqrt{2}/2} = \frac{3\pi}{4}$

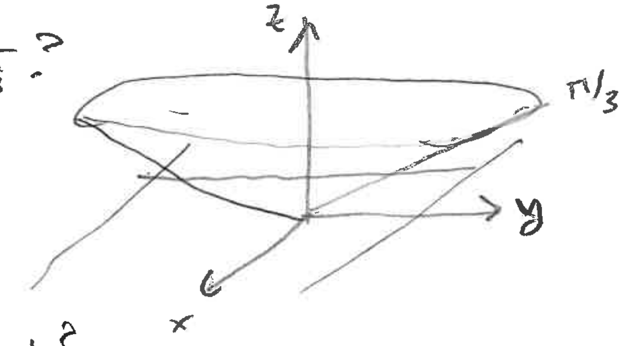
$\phi = \cos^{-1}\frac{-\sqrt{2}}{2} = \frac{3\pi}{4}$





What is the surface $\phi = \frac{\pi}{3}$?

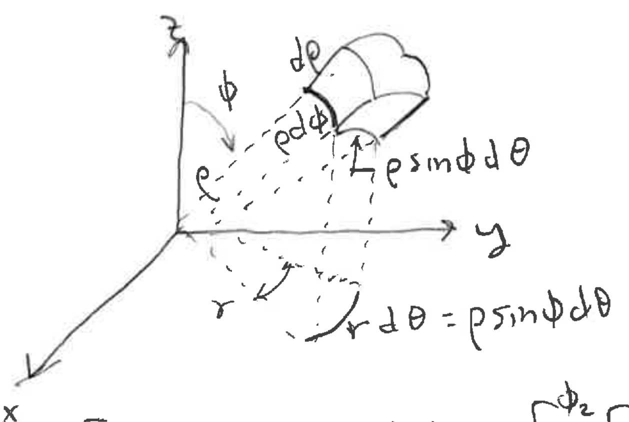
A half-cone in the upper plane.



What is the surface $\rho \cos\phi = 1$?

This is just $z = 1$. So a horizontal plane.

Integrals



$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\iiint_E f(x,y,z)\, dV = \int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

Like integration over a rectangle or a box, the simplest integral in spherical coordinates is a ball.

Example: Evaluate $\iiint_B (x^2+y^2+z^2)^2\, dV$, where B is a ball of radius 5 centered at origin.

Solution: $\displaystyle\int_0^{\pi}\int_0^{2\pi}\int_0^5 (\rho^2)^2 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

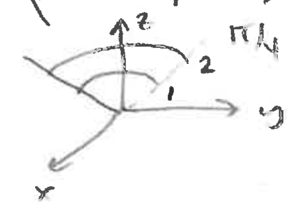
$\displaystyle = \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^5 \rho^6\, d\rho$

$\displaystyle = 2(2\pi)\frac{5^7}{7} =$

Example: Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\, dV$, where E lies above the cone $z = \sqrt{x^2+y^2}$ and between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$.

Solution: $z = \sqrt{x^2+y^2} \Rightarrow \rho\cos\phi = r = \rho\sin\phi \Rightarrow$

$\tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}$



$\rho^2 = 1 \Rightarrow \rho = 1$

$\rho^2 = 4 \Rightarrow \rho = 2$

$$\int_0^{\pi/4}\int_0^{2\pi}\int_1^2 \rho\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/4} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_1^2 \rho^3 d\rho$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right) 2\pi \left(\frac{2^4}{4} - \frac{1^4}{4}\right) = \frac{15\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$