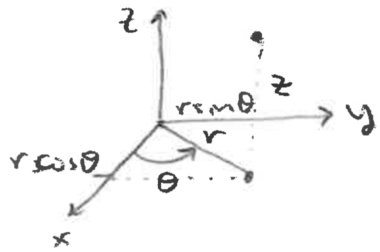
Tues 10/28

§15.7 Triple Integrals in Cylindrical Coordinates

Coordinate System



$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}\right\} \Rightarrow \begin{array}{l} r = \sqrt{x^2+y^2} \\ \frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{y}{x} \\ z = z \end{array}$$

Example: For $(r, \theta, z) = (4, \frac{\pi}{3}, -2)$, find (x, y, z).

$$x = r\cos\theta = 4\cos\frac{\pi}{3} = 4\left(\frac{1}{2}\right) = 2$$

$$y = r\sin\theta = 4\sin\frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$(x, y, z) = (2, 2\sqrt{3}, -2)$$

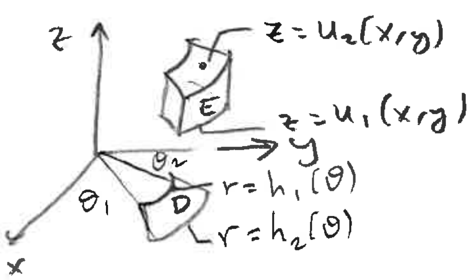
Example: For $(x, y, z) = (-2, 2\sqrt{3}, 3)$, find (r, θ, z).

$$r = \sqrt{x^2+y^2} = \sqrt{4+12} = 4$$

$$\theta = \tan^{-1}\frac{2\sqrt{3}}{-2} = \tan^{-1}\frac{\sqrt{3}/2}{-1/2} = \frac{2\pi}{3}$$

$$(r, \theta, z) = (4, \frac{2\pi}{3}, 3)$$

Integrals



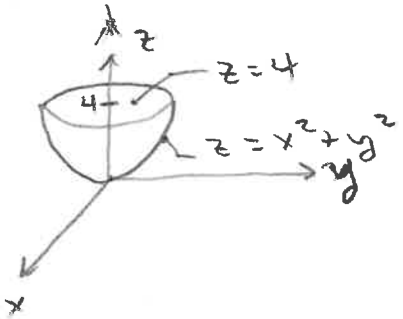
$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA$$

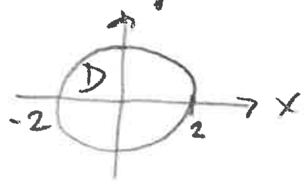
$$= \int_{\theta_1}^{\theta_2}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

$dV = r\,dz\,dr\,d\theta$

Example: Evaluate $\iiint_E z\,dV$, where E is enclosed by paraboloid $z = x^2+y^2$ and the plane $z=4$.

Solution:





$0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$z = 4 = x^2+y^2 = r^2 \Rightarrow r = 2$$

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 z\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{16-r^4}{2}\,r\,dr\,d\theta$$

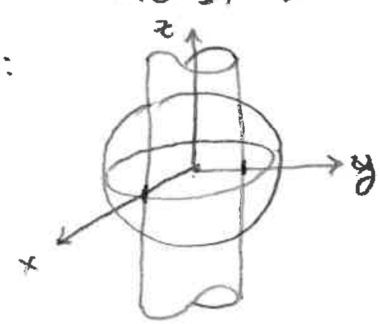
$$= \int_0^{2\pi}\int_0^2 \left(8r - \frac{r^5}{2}\right)dr\,d\theta$$

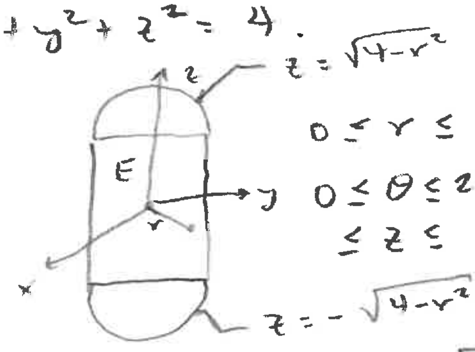
$$= \int_0^{2\pi}\left[4r^2 - \frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi}\left(16 - \frac{16}{3}\right)d\theta$$

$$= 2\pi\left(\frac{2}{3}\right)(16) = \frac{64\pi}{3}$$

Example: Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

Solution:





$0 \le r \le 1$
$0 \le \theta \le 2\pi$
$\le z \le$

$$x^2+y^2+z^2 = r^2+z^2 = 4 \Rightarrow z = \pm\sqrt{4-r^2}$$

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta = \int_0^{2\pi}\int_1^0 (4-r^2)^{1/2}\,d(4-r^2)\,d\theta$$

$$= \int_0^{2\pi}\frac{2}{3}(4-r^2)^{3/2}\Big|_1^0\,d\theta = 2\pi\left(\frac{2}{3}\right)\left(4^{3/2} - 3^{3/2}\right)$$

$$= \frac{4\pi}{3}\left(8 - 3\sqrt{3}\right)$$

Example: Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} x z \, dz \, dx \, dy$

Solution: $\sqrt{x^2+y^2} = r \le z \le 2$

$-\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \Rightarrow x^2+y^2 = r^2 = 4 \Rightarrow r = 2$

$-2 \le y \le 2 \Rightarrow r = 2 \qquad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^2\int_r^2 (r\cos\theta)\, z r \, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r^2\cos\theta\left(\frac{4-r^2}{2}\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \cos\theta\left(2r^2 - \frac{r^4}{2}\right) dr\, d\theta$$

$$= \overbrace{\int_0^{2\pi} \cos\theta \, d\theta}^{0} \int_0^2 \left(2r^2 - \frac{r^4}{2}\right) dr = 0$$