Alternatively,

$$\iiint_E \frac{z}{x^2+z^2}\,dV = \iint_D \frac{\pi}{4}\,dA = \int_1^4 \int_y^4 \frac{\pi}{4}\,dz\,dy$$

$$= \int_1^4 \frac{\pi}{4} z \Big|_y^4 dz = \int_1^4 \frac{\pi}{4}(4-y)\,dy$$

$$= \frac{\pi}{4}\left(4y - \frac{y^2}{2}\right)\Big|_1^4$$

$$= \frac{\pi}{4}\left(16 - 8 - 4 + \frac{1}{2}\right) = \frac{\pi}{4}\left(\frac{9}{2}\right) = \frac{9\pi}{8}$$

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Mon 10/27

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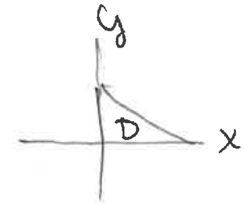
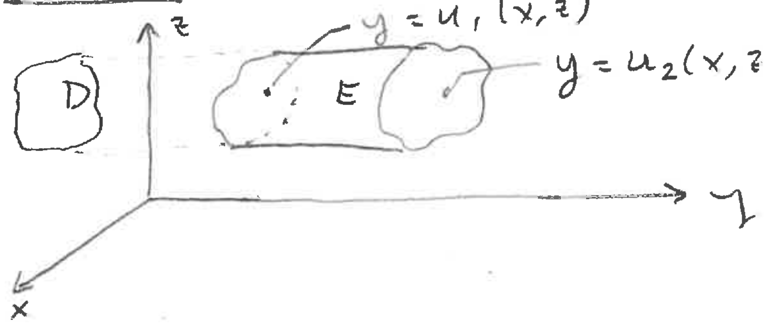
Quiz 9 Tuesday 10/28

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA$$

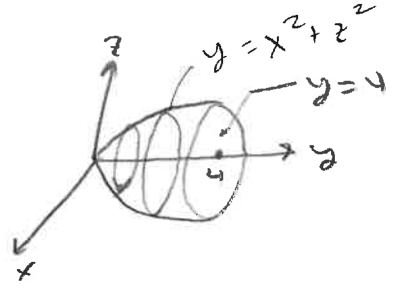


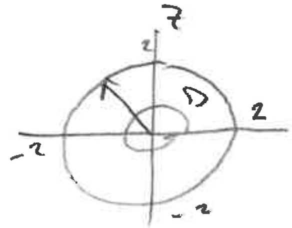

Type III




Example: Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is bounded by $y = x^2+z^2$ and the plane $y = 4$.

Solution:




$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$z = \pm\sqrt{4-x^2}$

$-2 \le x \le 2$

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dA$$

$$= \iint_D \sqrt{x^2+z^2}\, y \Big|_{x^2+z^2}^{4} dA = \iint_D \sqrt{x^2+z^2}\,(4-x^2-z^2)\, dA$$

Let $x = r\cos\theta$ and $z = r\sin\theta$, Then $x^2+z^2 = r^2$
$\Rightarrow \sqrt{x^2+z^2} = r$. Also $4-x^2-z^2 = 4-(x^2+z^2) = 4-r^2$,
$dA = r\,dr\,d\theta$.

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_0^{2\pi}\int_0^2 r(4-r^2)\, r\, dr\, d\theta = \int_0^2 (4r^2-r^4)\, dr \int_0^{2\pi} d\theta$$

$$= \frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2 \;\; \theta\Big|_0^{2\pi} = \left(\frac{32}{3} - \frac{32}{5}\right)2\pi = 64\pi\left(\frac{1}{3}-\frac{1}{5}\right)$$

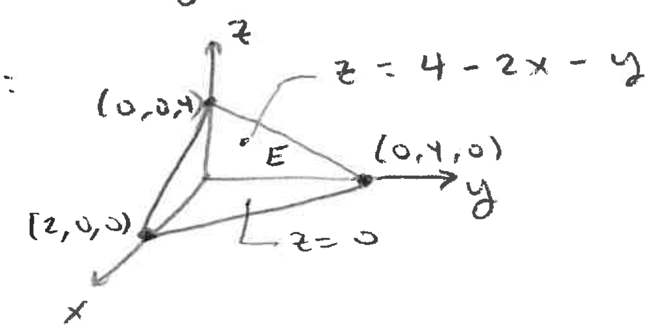
$$= \frac{128}{15}\pi$$

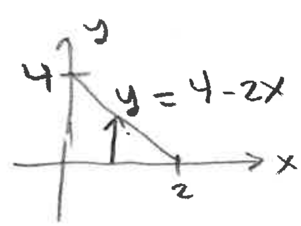
## Volume of a Solid

We saw that for $\iint_D f(x,y)\, dA$, that if $f(x,y)=1$,
then $\iint_D dA = A(D)$, the area of the region D.
Analogously, for $\iiint_E f(x,y,z)\, dV$, if $f(x,y,z)=1$,
then $\iiint_E dV = V(E)$, the volume of E.

Example: Find the volume of the solid region enclosed by the coordinate planes and the plane $2x+y+z=4$.

Solution:



$$V(E) = \iiint_E dV = \iint_D \int_0^{4-2x-y} dz\, dA = \iint_D (4-2x-y)\, dA$$

$$= \int_0^2 \int_0^{4-2x} (4-2x-y)\, dy\, dx = \int_0^2 \left[4y - 2xy - \frac{y^2}{2}\right]_0^{4-2x} dx$$

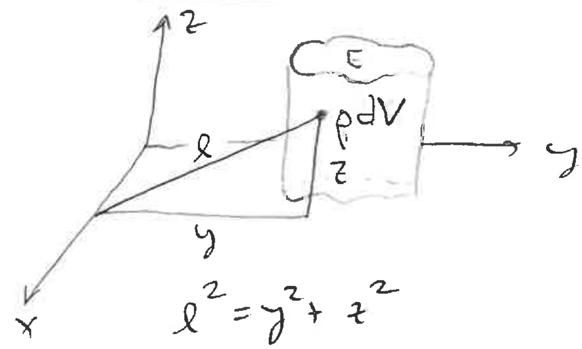
$$= \int_0^2 [16 - 8x - 8x + 4x^2 - 8 + 8x - 2x^2)\, dx$$

$$= \int_0^2 (8 - 8x + 2x^2)\, dx = 8x - 4x^2 + \frac{2}{3}x^3 \Big|_0^2$$

$$= 16 - 16 + \frac{16}{3} = \frac{16}{3}$$

<u>Moments of Inertia</u>

the density $\rho$ has units mass per volume



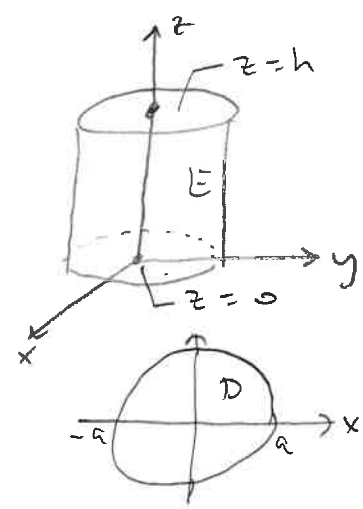

$$\ell^2 = y^2 + z^2$$

$$I_x = \iiint_E (y^2+z^2)\rho\, dV$$

$$I_y = \iiint_E (x^2+z^2)\rho\, dV$$

$$I_z = \iiint_E (x^2+y^2)\rho\, dV$$

Example: Find $I_z$ of the solid cylinder $x^2+y^2 \le a^2$, $0 \le z \le h$, constant $\rho$.




Solution: $I_z = \iiint_E (x^2+y^2)\rho\, dV$

$$= \iint_D \int_0^h (x^2+y^2)\rho\, dz\, dA$$

$$= \iint_D \rho h (x^2+y^2)\, dA$$

$$= \rho h \int_0^{2\pi}\int_0^a r^2\, r\, dr\, d\theta$$

$$= \rho h \int_0^{2\pi} \frac{r^4}{4}\Big|_0^a d\theta$$

$$= 2\pi \rho h \frac{a^4}{4} = \frac{\pi \rho h a^4}{2}$$