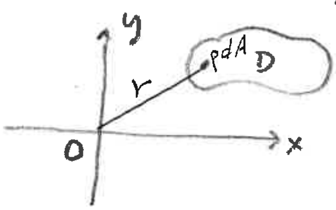
Fri 10/24

HW 15.4 Problem

$I_0 = I_x + I_y$

where I_0 is the moment of inertia about the origin, also called polar moment of inertia.



$$I_0 = \iint_D r^2 \rho \, dA = \iint_D (x^2+y^2)\rho \, dA$$
$$= \iint_D x^2 \rho \, dA + \iint_D y^2 \rho \, dA$$
$$= I_x + I_y$$

§15.6 Triple Integrals

Analogous to a double integral over a rectangle R, $\iint_R f(x,y)\, dA$, we have a triple integral over a box B, $\iiint_B f(x,y,z)\, dV$.

By Fubini's theorem, we write this as an iterated integral in any order. So there are six ways to evaluate the integral.

Example: Evaluate $\iiint_B (xy+z^2)\, dV$, where $B = \{(x,y,z) \mid 0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3\}$.

Solution: $$\iiint_B (xy+z^2\, dV = \int_0^2\int_0^1\int_0^3 (xy+z^2)\, dz\, dy\, dx$$
$$= \int_0^2 \int_0^1 \left[xyz + \frac{z^3}{3}\right]_0^3 dy\, dx$$
$$= \int_0^2 \int_0^1 (3xy + 9)\, dy\, dx$$
$$= \int_0^2 \left[\frac{3}{2}xy^2 + 9y\right]_0^1 dx$$
$$= \int_0^2 \left(\frac{3}{2}x + 9\right) dx = \frac{3}{4}x^2 + 9x \Big)_0^2$$
$$= 3 + 18 = 21$$

Also, we could do

$$\iiint_B (xy+z^2)\,dV = \int_0^3\int_0^1\int_0^2 (xy+z^2)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^1 \left[\frac{x^2y}{2} + xz^2\right]_0^2 dy\,dz$$

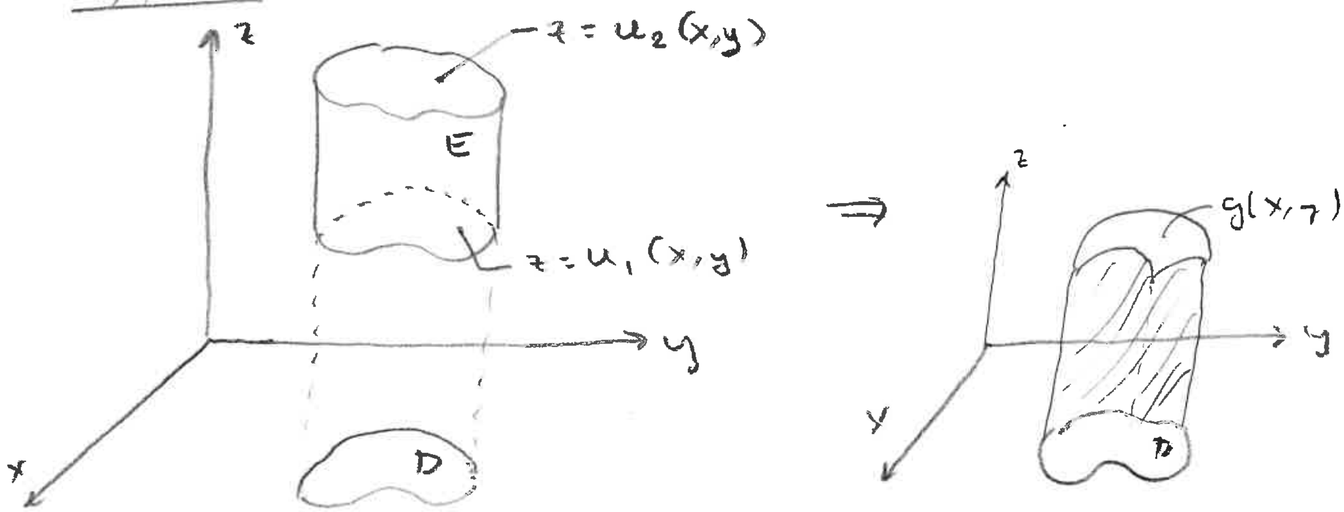
$$= \int_0^3\int_0^1 (2y+2z^2)\,dy\,dz = \int_0^3 \left[y^2+2yz^2\right]_0^1 dz$$

$$= \int_0^3 (1+2z^2)\,dz = z + \frac{2}{3}z^3 \Big|_0^3 = 3+18 = 21$$

Analogous to a double integral over a region D, $\iint_D f(x,y)\,dA$, we now have a triple integral over a solid E; $\iiint_E f(x,y,z)\,dV$.

We have three types corresponding to each of the three coordinates.

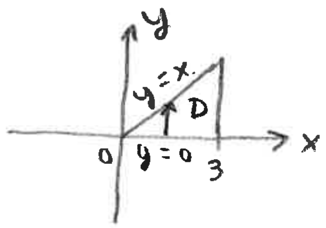
Type I



$$\iiint_E f(x,y,z)\,dV = \iint_D \underbrace{\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right]}_{g(x,y)} dA$$

Example: Evaluate $\iiint_E y \, dV$, where $E = \{(x,y,z) \mid 0 \le x \le 3,\ 0 \le y \le x,\ x-y \le z \le x+y\}$

Solution: We will use a Type I integral

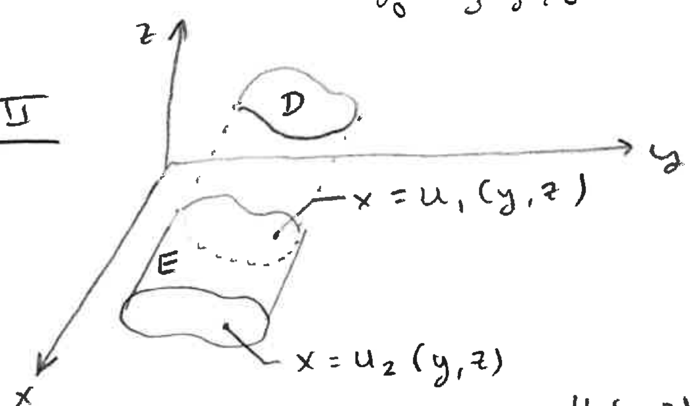


$$\iiint_E y \, dV = \iint_D \int_{x-y}^{x+y} y \, dz \, dA = \iint_D y(x+y-x+y) \, dA$$

$$= \iint_D 2y^2 \, dA = \int_0^3 \int_0^x 2y^2 \, dy \, dx$$

$$= \int_0^3 \frac{2}{3} y^3 \Big|_0^x dx = \frac{2}{3} \int_0^3 x^3 \, dx = \frac{2}{3} \frac{x^4}{4} \Big|_0^3 = \frac{27}{2}$$

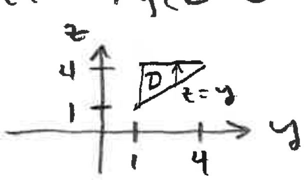
Type II



$$\iiint_E f(x,y,z) \, dV = \iint_D \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z) \, dx \, dA$$

Example: Evaluate $\iiint_E \frac{z}{x^2+z^2} \, dV$, where $E = \{(x,y,z) \mid 1 \le y \le 4,\ y \le z \le 4,\ 0 \le x \le z\}$.

Solution: Type II integral



$$\iiint_E \frac{z}{x^2+z^2} \, dV = \iint_D \int_0^z \frac{z}{x^2+z^2} \, dx \, dA = \iiint_D \int_0^z \frac{z}{z^2} \frac{1}{\left(\frac{x}{z}\right)^2+1} (z) \, d\!\left(\frac{x}{z}\right) dA$$

$$= \iiint_D \int_0^z \frac{1}{\left(\frac{x}{z}\right)^2+1} \, d\!\left(\frac{x}{z}\right) dA = \iint_D \tan^{-1}\left(\frac{x}{z}\right) \Big|_0^z \, dA$$

$$= \iint_D \frac{\pi}{4} \, dA = \frac{\pi}{4} \iint_D dA = \frac{\pi}{4} A(D) = \frac{\pi}{4} \frac{1}{2}(3)3 = \frac{9}{8}\pi$$

Alternatively,

$$\iiint_E \frac{z}{x^2+z^2}\,dV = \iint_D \frac{\pi}{4}\,dA = \int_1^4 \int_y^4 \frac{\pi}{4}\,dz\,dy$$

$$= \int_1^4 \frac{\pi}{4} z \Big|_y^4 dz = \int_1^4 \frac{\pi}{4}(4-y)\,dy$$

$$= \frac{\pi}{4}\left(4y - \frac{y^2}{2}\right)\Big|_1^4$$

$$= \frac{\pi}{4}\left(16 - 8 - 4 + \frac{1}{2}\right) = \frac{\pi}{4}\left(\frac{9}{2}\right) = \frac{9\pi}{8}$$