Wed 10/21

§15.4 Applications of Double Integrals
- Moments and centers of mass
- Moment of Inertia and radii gyration

mass: $m = \iint_D \rho \, dA$

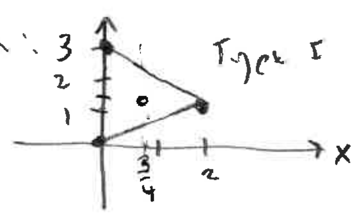
moment about x-axis: $M_x = \iint_D y\rho \, dA$

moment about y-axis: $M_y = \iint_D x\rho \, dA$

Center of mass: $\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$

Example: Find the mass and center of mass of the lamina that occupies the triangular region with vertices $(0,0)$, $(2,1)$, and $(0,3)$, with density $\rho = x + y$.

Solution:



Type I

$y_1 = g_1(x) = -x + 3$

$y_2 = g_2(x) = \frac{x}{2}$

$$m = \iint_D \rho \, dA = \int_0^2 \int_{y_1}^{y_2} (x+y) \, dy \, dx = \int_0^2 \int_{\frac{x}{2}}^{3-x} (x+y) \, dy \, dx$$

$$= \int_0^2 \left[xy + \frac{y^2}{2} \right]_{\frac{x}{2}}^{3-x} dx$$

$$= \int_0^2 \left(3x - x^2 + \frac{9}{2} - 3x + \frac{x^2}{2} - \frac{x^2}{2} - \frac{x^2}{8} \right) dx$$

$$= \int_0^2 \left(-\frac{9}{8}x^2 + \frac{9}{2} \right) dx = -\frac{3}{8}x^3 + \frac{9}{2}x \Big)_0^2$$

$$= -3 + 9 = 6$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{6}\iint_D x\rho\,dA = \frac{1}{6}\int_0^2\int_{\frac{x}{2}}^{3-x}(x^2+xy)\,dy\,dx$$

$$= \frac{1}{6}\int_0^2\left[x^2y + \frac{xy^2}{2}\right]_{\frac{x}{2}}^{3-x}dx$$

$$= \frac{1}{6}\int_0^2\left(3x^2 - x^3 + \frac{x}{2}(9-6x+x^2) - \frac{x^3}{2} - \frac{x^3}{8}\right)dx$$

$$= \frac{1}{6}\int_0^2\left(3x^2 - x^3 + \frac{9}{2}x - 3x^2 + \frac{x^3}{2} - \frac{x^3}{2} - \frac{x^3}{8}\right)dx$$

$$= \frac{1}{6}\int_0^2\left(-\frac{9}{8}x^3 + \frac{9}{2}x\right)dx$$

$$= \frac{1}{6}\left[-\frac{9x^4}{32} + \frac{9x^2}{4}\right]_0^2 = \frac{1}{6}\left(-\frac{9}{2}+9\right) = \frac{3}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{6}\iint y\rho\,dA = \frac{1}{6}\int_0^2\int_{\frac{x}{2}}^{3-x}(xy+y^2)\,dy\,dx$$

$$= \frac{1}{6}\int_0^2\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{\frac{x}{2}}^{3-x}dx$$

$$= \frac{1}{6}\int_0^2\left(-\frac{9}{2}x + 9\right)dx = \frac{1}{6}\left(-\frac{9x^2}{4} + 9x\right)\Big|_0^2$$

$$= \frac{1}{6}(-9+18) = \frac{3}{2}$$

Moment of Inertia

Also called the second moment, is the moment of a moment.

$$I_x = \iint_D y^2\rho\,dA$$

$$I_y = \iint_D x^2\rho\,dA$$

The radius of gyration $\bar{\bar{x}}$ with respect to y axis is

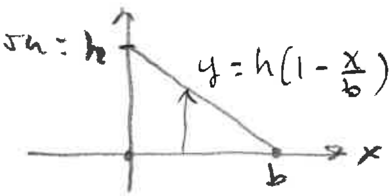
$$I_y = \iint_D x^2\rho\,dA = \iint_D \bar{\bar{x}}^2\rho\,dA = \bar{\bar{x}}^2\iint_D\rho\,dA = \bar{\bar{x}}^2 m \Rightarrow \bar{\bar{x}} = \sqrt{\frac{I_y}{m}}$$

Similarly,

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}$$

Example: A lamina with constant density ρ occupies the triangle with vertices $(0,0)$, $(b,0)$, $(0,h)$. Find the moments of inertia I_x and I_y and the radii of gyration $\bar{\bar{x}}$ and $\bar{\bar{y}}$.

Solution:



$$I_x = \rho \iint_D y^2 \, dA = \rho \int_0^b \int_0^{h(1-\frac{x}{b})} y^2 \, dy \, dx$$

$$= \rho \int_0^b \frac{y^3}{3} \Big|_0^{h(1-\frac{x}{b})} dx = \frac{\rho h^3}{3} \int_0^b \left(1 - \frac{x}{b}\right)^3 dx$$

$$= -b \frac{\rho h^3}{3} \int_0^b \left(1 - \frac{x}{b}\right)^3 d\left(1 - \frac{x}{b}\right)$$

$$= -\frac{\rho b h^3}{3} \frac{\left(1 - \frac{x}{b}\right)^4}{4} \Big|_0^b = \frac{\rho b h^3}{12}$$

$$I_y = \rho \int_0^b \int_0^{h(1-\frac{x}{b})} x^2 \, dy \, dx$$

$$= \rho \int_0^b x^2 y \Big|_0^{h(1-\frac{x}{b})} dx = \rho \int_0^b h\left(x^2 - \frac{x^3}{b}\right) dx$$

$$= \rho h \left[\frac{x^3}{3} - \frac{x^4}{4b} \right]_0^b = \rho h \left(\frac{b^3}{3} - \frac{b^3}{4} \right) = \frac{\rho h b^3}{12}$$

$$m = \iint_D \rho \, dA = \rho \iint_D dA = \rho A(D) = \rho \frac{bh}{2}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{\rho h b^3}{12} \frac{2}{\rho b h}} = \sqrt{\frac{b^2}{6}} = \frac{b}{\sqrt{6}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\rho b h^3}{12} \frac{2}{\rho b h}} = \sqrt{\frac{h^2}{6}} = \frac{h}{\sqrt{6}}$$