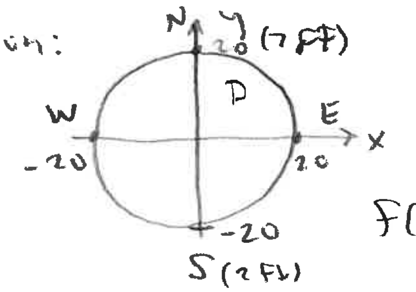
Tues 10/21

§15.3 Double Integrals in Polar Coordinates

Application: Find the volume of H_2O in a swimming pool with a 40 ft diameter and a bottom that is constant from east to west and increases from 2 ft on the south to 7 ft on the north.

Solution:



$$\iint_D f(x,y)\,dA$$

$$f(x,y) = \frac{5}{40}(y+20) + 2$$

$$f(x,-20) = 2$$

$$f(x,20) = 5+2 = 7$$

$$f(x,y) = \frac{1}{8}y + \frac{5}{2} + 2 = \frac{1}{8}y + \frac{9}{2}$$

$$\iint_D \left(\frac{1}{8}y + \frac{9}{2}\right) dA = \int_0^{2\pi}\int_0^{20}\left(\frac{1}{8}r\sin\theta + \frac{9}{2}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{20}\left(\frac{1}{8}r^2\sin\theta + \frac{9}{2}r\right) dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^3}{24}\sin\theta + \frac{9r^2}{4}\right)\Big|_0^{20} d\theta$$

$$= \int_0^{2\pi}\left[400\left(\frac{5}{6}\right)\sin\theta + \frac{9}{4}(400)\right] d\theta$$

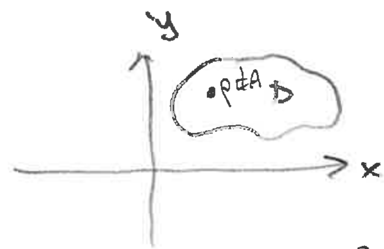
$$= \cancel{400\left(\frac{5}{6}\right)(-\cos\theta)} + 900\,\theta\Big|_0^{2\pi}$$

$$= 1800\pi \text{ ft}^3$$

§15.4 Double Integral Applications

The mass density ρ has units mass per area. A point mass at (x, y) is $\rho\, dA$. The total mass m of a thin plate (called a lamina) that occupies a region D is

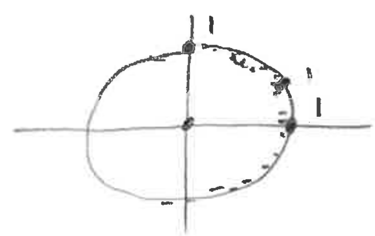
$$m = \iint_D \rho\, dA$$



The same formula applies for charge density σ, which has units C/m^2. the total charge on the lamina is

$$\sigma = \iint_D \sigma\, dA$$

Example: Electric charge is distributed over a disk $x^2 + y^2 \le 1$ m^2 so that the charge density is $\sigma = \sqrt{x^2 + y^2}$ C/m^2.



Let $r = \sqrt{x^2 + y^2}$.

Or $\sigma = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$
$= r$

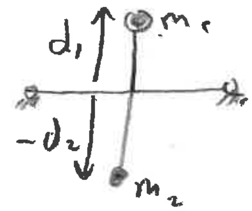
$x = \frac{\sqrt{3}}{2}$ $\quad \sigma = \sqrt{\frac{3}{4} + \frac{1}{4}}$
$y = \frac{1}{2}$ $\quad = 1$

Find the total charge on the disk.

Solution: $Q = \iint_D \sigma\, dA = \int_0^{2\pi}\int_0^1 r\, r\, dr\, d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big|_0^1 d\theta$

$= \frac{1}{3}\int_0^{2\pi} d\theta = \frac{1}{3}\theta\Big|_0^{2\pi} = \frac{2}{3}\pi$ C

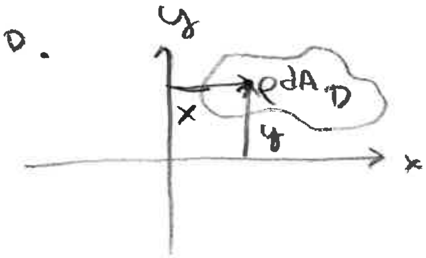
Moments and Centers of Mass



A moment is the product of some quantity (area, mass, force) and a signed distance. We saw that moment of force is called torque.

$m_1 d_1 + m(-d_2) = 0$
then it is stationary

We will do moment of mass of a lamina that occupies region D.



The moment about the x axis

$$M_x = \iint_D y\rho \, dA$$

The moment about the y-axis is

$$M_y = \iint_D x\rho \, dA.$$

Now suppose we hold x and y constant, then these will be the location $(\bar{x}, \bar{y})$ of the center of mass.

$$M_x = \iint_D y\rho \, dA = \iint_D \bar{y}\rho \, dA = \bar{y}\iint_D \rho \, dA = \bar{y}m$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_D y\rho \, dA}{\iint_D \rho \, dA}$$

$$M_y = \iint x\rho \, dA = \iint_D \bar{x}\rho \, dA = \bar{x}\iint_D \rho \, dA = \bar{x}m$$

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\rho \, dA}{\iint_D \rho \, dA}$$