Mon 10/20

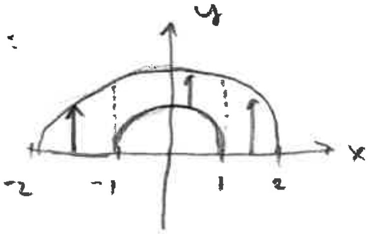
HW 14.7 due Tuesday 10/21 at 11:59 p.m.

Quiz 8 Tuesday 10/21 at 4:50.

§15.3 Double Integrals in Polar Coordinates

Example: Let $f(x,y) = 3x + 4y^2$ and region D to be the upper half plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$. Find $\iint_D f(x,y)\,dA$.

Solution:

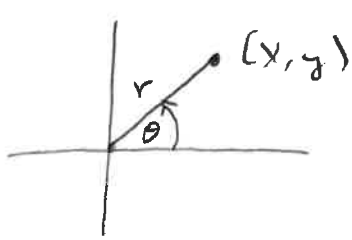


Type I

$$\iint_D f(x,y)\,dA = \int_{-2}^{-1}\int_0^{\sqrt{4-y^2}} f(x,y)\,dy\,dx + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-y^2}} f(x,y)\,dy\,dx$$
$$+ \int_1^2\int_0^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

This is a mess! It will involve trig substitution. We will transform to polar coordinates.

Polar Coordinates

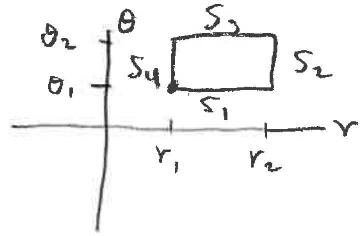


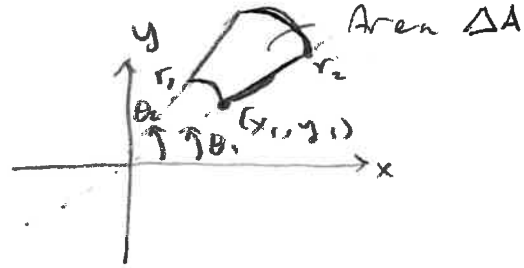
$x = r\cos\theta = g(r,\theta)$
$y = r\sin\theta = h(r,\theta)$

$$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,dA$$

↑ what is this in terms of r and θ?

Consider a polar rectangle





For side s_1, $\theta = \theta_1$ a constant.

$x = r\cos\theta_1$ and $y = r\sin\theta_1$. Dividing we have

$$\frac{y}{x} = \frac{\sin\theta_1}{\cos\theta_1} \Rightarrow y = \underbrace{(\tan\theta_1)}_{\substack{\text{slope}\\ \text{constant}}}x + \underset{\uparrow\ \text{y-intercept}}{0}$$

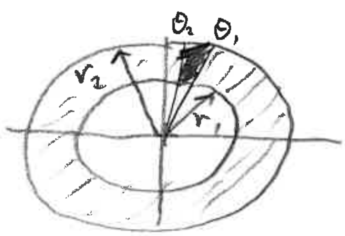
Let $x_1 = r_1\cos\theta_1$ and $y_1 = r_1\sin\theta_1$

For side s_2, $r = r_2$ is constant, $x = r_2\cos\theta$ and $y = r_2\sin\theta$. or $x^2 = r_2^2\cos^2\theta$ and $y^2 = r_2^2\sin^2\theta$.
Adding, we get $x^2 + y^2 = r_2^2$ a circle with radius r_2

For side s_3, $y = (\tan\theta_2)x$, $\theta_2 > \theta_1$

For side s_4, $x^2 + y^2 = r_1^2$

The area of the annulus is

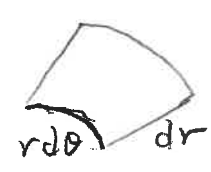


$$\text{Area} = \pi(r_2^2 - r_1^2)$$

The area of the shaded region is

$$\text{Area} = \pi(r_2^2 - r_1^2)\frac{\theta_2 - \theta_1}{2\pi} = \frac{(r_2+r_1)(r_2-r_1)(\theta_2-\theta_1)}{2}$$

$$\Delta A = \bar{r}\,\Delta r\,\Delta\theta$$



Let $\Delta r, \Delta\theta \to 0$, then

$$dA = r\,dr\,d\theta$$

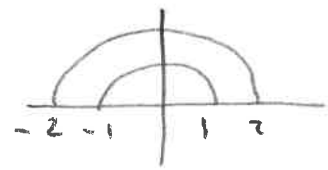
$dA = r\,dr\,d\theta$

In other words, double integral in polar coordinates is defined as

$$\iint_D f(x,y)\,dA = \lim_{n\to\infty}\sum_{i=1}^{n}\lim_{m\to\infty}\sum_{j=1}^{m} f(r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)\,\bar{r}\,\Delta r\,\Delta\theta$$

$$= \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example: $f(x,y) = 3x + 4y^2$



Solution: $\displaystyle\int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$

$$= \int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi}\Big[r^3\cos\theta + r^4\sin^2\theta\Big]_1^2\, d\theta$$

$$= \int_0^{\pi}(7\cos\theta + 15\sin^2\theta)\,d\theta$$

$$= \cancelto{0}{7\sin\theta\Big|_0^{\pi}} + 15\int_0^{\pi}\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right)d\theta$$

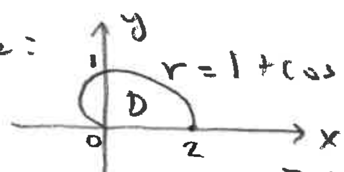
$$= 15\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{15}{2}\pi$$

Quiz 8 is a special case to find area of region.

$f(x,y) = 1$, $r_1 = 0$, $r_2 = h(\theta)$.

$$A(D) = \iint_D dA = \int_{\theta_1}^{\theta_2}\int_0^{h(\theta)} r\,dr\,d\theta$$

Example:



Find area D.

Solution: $\displaystyle A(D) = \int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi} \frac{r^2}{2}\Big|_0^{1+\cos\theta} d\theta$

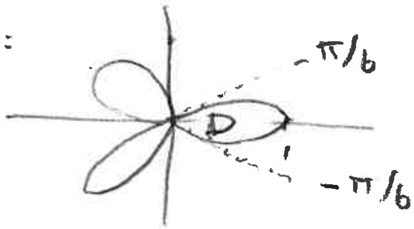
$$= \frac{1}{2}\int_0^{\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{3}{4}\pi$$

Example: Calculate the area of one petal of a rose described by $r = \cos 3\theta$.

Solution:



$$0 = \cos 3\theta \Rightarrow$$
$$\pm \frac{\pi}{2} = 3\theta \Rightarrow \theta = \pm \frac{\pi}{6}$$

$$A(D) = \int_{-\frac{\pi}{6}}^{\pi/6} \int_0^{\cos 3\theta} r \, dr \, d\theta = \int_{-\frac{\pi}{6}}^{\pi/6} \frac{r^2}{2} \Big|_0^{\cos 3\theta} d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\pi/6} \frac{\cos^2 3\theta}{2} \, d\theta = \frac{1}{2} \int_{-\frac{\pi}{6}}^{\pi/6} \left(\frac{1}{2} + \frac{\cos 6\theta}{2} \right) d\theta$$

$$= \frac{1}{2} \left[\frac{\theta}{2} + \frac{\sin 6\theta}{12} \right]_{-\frac{\pi}{6}}^{\pi/6} = \frac{1}{2} \left(\frac{\pi}{12} + \frac{\pi}{12} \right) = \frac{\pi}{12}$$