Fri 10/17

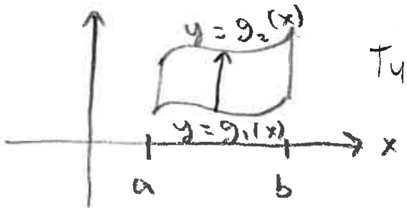
§15.2 Double Integrals over General Regions

- Type I Regions
- Type II Regions
- Properties of Double Integrals
- Examples

HW 15.2 due Monday 10/27.

Type I Region

A type I region is one bounded by $x=a$, $x=b$, $y=g_1(x)$, and $y=g_2(x)$.

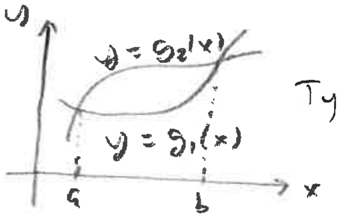


Type I

Sometimes, we need to find a and b if $g_1(x)$ and $g_2(x)$ intersect

$g_1(x)=g_2(x) \Rightarrow x=a,b$

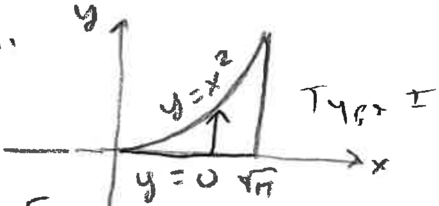
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



Type I

Example: Find $\iint_D x\cos y\,dA$ for D bounded by $y=0$, $y=x^2$, and $x=\sqrt{\pi}$.

Solution:



Type I

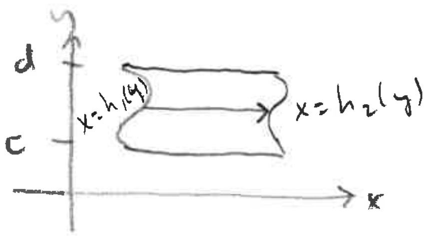
$$\int_0^{\sqrt{\pi}}\int_0^{x^2} x\cos y\,dy\,dx = \int_0^{\sqrt{\pi}} x\sin y\Big|_0^{x^2} dx$$

$$= \int_0^{\sqrt{\pi}} (x\sin x^2 - x\sin 0)\,dx = \int_0^{\sqrt{\pi}} x\sin x^2\,dx$$

$$= \frac{1}{2}\int_0^{\sqrt{\pi}} \sin x^2\,dx^2 = \frac{1}{2}(-\cos x^2)\Big|_0^{\sqrt{\pi}} = \frac{1}{2}(1+1) =$$

Type II Region

A type II region D is bounded by $y=c$, $y=d$, $x=h_1(y)$, and $x=h_2(y)$.

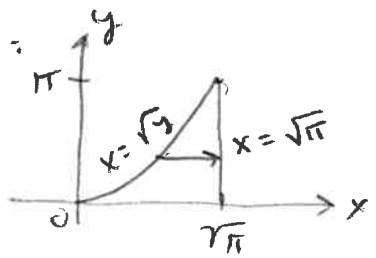


Type II

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example: Rework the previous example as a Type II region.

Solution:



$$\sqrt{y} = \sqrt{\pi} \Rightarrow y = \pi$$

$$\int_0^{\pi}\int_{\sqrt{y}}^{\sqrt{\pi}} x\cos y\,dx\,dy = \int_0^{\pi} \frac{x^2}{2}\cos y \Big|_{\sqrt{y}}^{\sqrt{\pi}} dy$$

$$= \int_0^{\pi}\left(\frac{\pi}{2}\cos y - \frac{y}{2}\cos y\right)dy$$

$$= \frac{\pi}{2}\int_0^{\pi}\cos y\,dy - \frac{1}{2}\int_0^{\pi} y\cos y\,dy$$

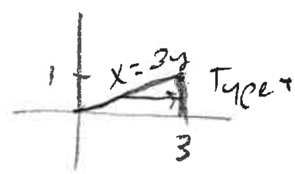
$$= \frac{\pi}{2}\sin y\Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi} y\,d\sin y$$

$$= -\frac{1}{2}\left[\, y\sin y\Big|_0^{\pi} - \int_0^{\pi}\sin y\,dy \right]$$

$$= \frac{1}{2}(-\cos y)\Big|_0^{\pi} = \frac{1}{2}(-(-1)+1) = 1$$

In this case, we could switch the order. Sometimes, we are not able to switch.

Example: $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\,dy\,dx$



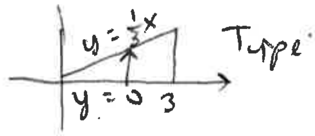
$= \int_0^3 y e^{x^2} \Big|_0^{\frac{x}{3}} dx$

$x = 3y \Rightarrow$

$y = \frac{1}{3}x$

$= \int_0^3 \frac{x}{3} e^{x^2}\,dx = \frac{1}{6}\int_0^3 e^{x^2}\,dx^2$

$= \frac{1}{6} e^{x^2} \Big|_0^3 = = \frac{1}{6}(e^9 - 1)$



Properties of Double Integrals

$\iint_D [f(x,y) + g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

$\iint_D c f(x,y)\,dA = c \iint_D f(x,y)\,dA$

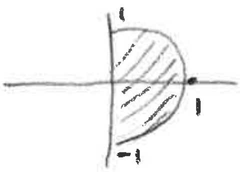
For $g(x,y) \le f(x,y) \le h(x,y)$ on D,

$\iint_D g(x,y)\,dA \le \iint_D f(x,y)\,dA \le \iint_D h(x,y)\,dA$

Special case: $g(x,y) = m$ and $h(x,y) = M$.

$m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D)$

Example: Estimate $\iint_D \sqrt{4 - x^2y^2}\,dA$, $D = \{(x,y) \mid x^2+y^2 \le 1, x \geq$



$x^2y^2 \ge 0 \Rightarrow -x^2y^2 \le 0 \Rightarrow \sqrt{4 - x^2y^2} \le \sqrt{4} = 2$

$M = 2$

$x^2y^2 \le 1 \Rightarrow -x^2y^2 \ge -1 \Rightarrow \sqrt{4 - x^2y^2} \ge \sqrt{3} = m$

$0.866\pi \approx \frac{\sqrt{3}}{2}\pi \le \iint_D \sqrt{4 - x^2y^2}\,dA \le \pi$