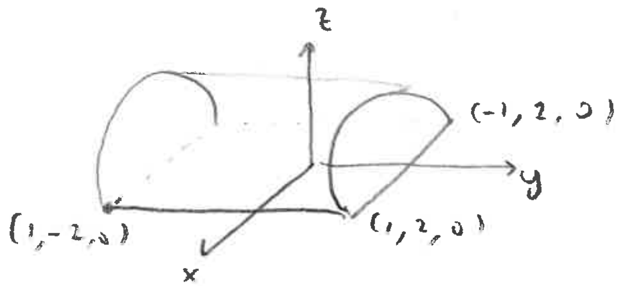
Wed 10/15

§15.1 Double Integrals over Rectangles

Last time, we saw that

$$\iint_R \sqrt{1-x^2}\,dx = 2\pi \quad \text{for } R = [-1,1] \times [-2,2].$$

Let $z = \sqrt{1-x^2}$. then $x^2 + z^2 = 1$, $z \geq 0$.



$$\text{Volume} = \frac{\pi(1)^2}{2}(4) = 2\pi$$

<u>Iterated Integrals</u>

Fubini theorem says if $f(x,y)$ is continuous on R, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Let's rework the example.

$$\int_{-2}^{2}\int_{-1}^{1} \sqrt{1-x^2}\,dx\,dy = \int_{-2}^{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta\,dy$$

let $x = \sin\theta$. then $dx = \cos\theta\,d\theta$.

$x = -1 \Rightarrow \theta = \sin^{-1}(-1) = -\frac{\pi}{2}$

$x = 1 \Rightarrow \theta = \sin^{-1} 1 = \frac{\pi}{2}$

$$\int_{-2}^{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\,\cos\theta\,d\theta\,dy = \int_{-2}^{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\theta|\cos\theta\,d\theta\,dy$$

$$= \int_{-2}^{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta\,dy = \int_{-2}^{2}\int_{-\frac{\pi}{2}}^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta\,dy$$

$$= \int_{-2}^{2} \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{\pi/2} dy = \int_{-2}^{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)dy = \frac{\pi}{2}\int_{-2}^{2} dy$$

$$= \frac{\pi}{2} y \Big)_{-2}^{2} = \frac{\pi}{2}(4) = 2\pi$$

We can switch the order.

$$\int_{-1}^{1}\int_{-2}^{2}\sqrt{1-x^2}\,dy\,dx = \int_{-1}^{1}\sqrt{1-x^2}\int_{-2}^{2}dy\,dx = 4\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

$$= 4\left(\frac{\pi}{2}\right) = 2\pi$$

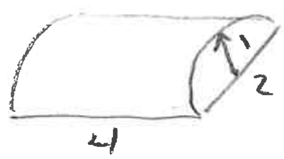
We can separate the integrals.

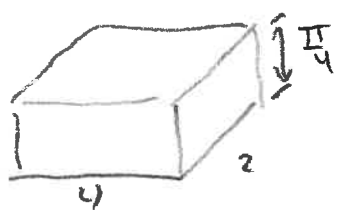
$$\int_{-2}^{2}\underbrace{\int_{-1}^{1}\sqrt{1-x^2}\,dx}_{\text{is a number}}\,dy = \int_{-1}^{1}\sqrt{1-x^2}\,dx\int_{-2}^{2}dy$$

Special case

If $f(x,y) = g(x)\,h(y)$, then

$$\iint_R f(x,y)\,dA = \int_a^b g(x)\,dx\int_c^d h(y)\,dy$$





$V_1 = 2\pi = V_2 = Az$

$z = \frac{2\pi}{8} = \frac{\pi}{4}$

This is the average value of $f(x,y)$.

$$\iint_R f(x,y)\,dA = \iint_R f_{avg}\,dA = f_{avg}\iint_R dA = f_{avg}\,A(R)$$

$$f_{avg} = \frac{\iint_R f(x,y)\,dA}{A(R)}$$, where $A(R)$ is area of rectangle.

$$S_1 = \frac{\pi(1)^2}{2}(2) + \frac{2\pi(1)}{2}4 = \pi + 4\pi = 5\pi = 3\pi + 2\pi$$

$$S_2 = 2\left(\frac{\pi}{4}\right)2 + 4\left(\frac{\pi}{4}\right)2 + 8 = \pi + 2\pi + 8 = 3\pi + 8$$

$8 > 2\pi \approx 6.28$