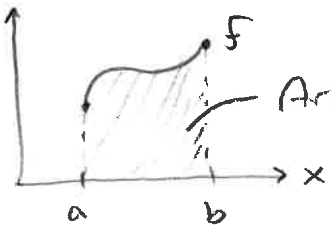
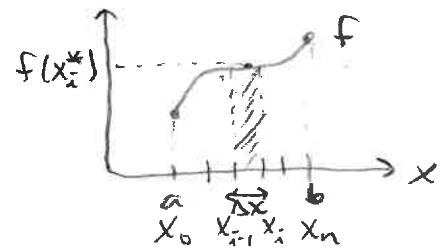
Tues 10/14

§15.1 Double Integrals over Rectangles

Recall for $f(x) \geq 0$, the integral over an interval $[a,b]$ is the area under the curve.





We can approximate the integral with a Riemann sum. We divide the interval into n subintervals.

$$\frac{b-a}{n} = \Delta x = x_i - x_{i-1}, \quad \text{where } i = 1, 2, \ldots, n.$$

Pick a point $x_i^* \in [x_{i-1}, x_i]$ to find the height $f(x_i^*)$. Then the area of the i^{th} rectangle is $f(x_i^*)\Delta x$. Note the midpoint is $x_i^* = \frac{x_i + x_{i-1}}{2}$.

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i^*)\Delta x.$$ To get the exact area,

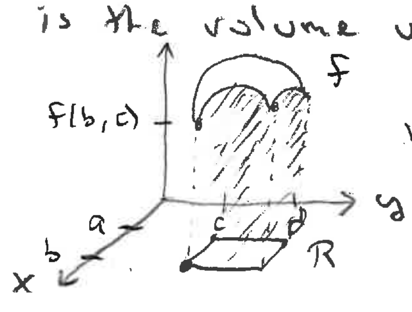
we take the limit as $n \to \infty$, which means $\Delta x \to 0$.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

Let's extend this to two dimensions. Let $f(x,y) \geq 0$, then the integral over a rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$$

is the volume under the surface.

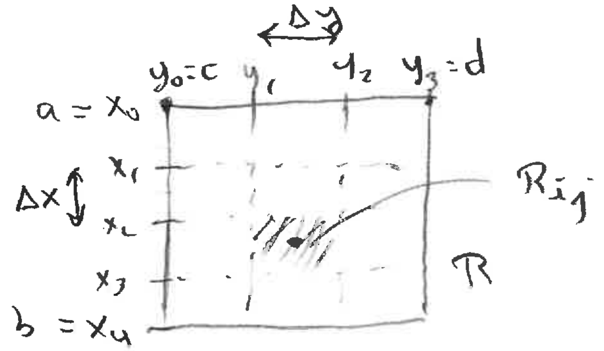


volume of a solid

We can approximate this volume with a Riemann sum. First divide each interval into subintervals,

$\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{m}$. In other words,

for $m=3$ and $n=4$, we have $mn = 12$ subrectangles R_{ij} where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.



We clearly see that $\Delta A = \Delta x \Delta y$. We need a height, so pick a point, (x_i^*, y_j^*).

We could use the mid points $\left(\frac{x_i + x_{i-1}}{2}, \frac{y_j + y_{j-1}}{2}\right)$ or the upper right points $(x_i^*, x_j^*) = (x_i, y_j)$.

Then the volume of the ij^{th} rectangle is

Volume of ij^{th} rectangle $\approx f(x_i^*, y_j^*)\Delta A$. The total volume is

$$\text{Volume} \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta A.$$

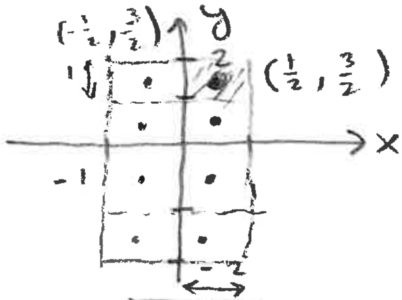
The total volume is

$$\text{Volume} = \lim_{n\to\infty}\sum_{i=1}^{n}\lim_{m\to\infty}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta A$$
$$= \iint_R f(x,y)\,dA$$

Example: Approximate $\iint_R f(x,y)\,dA$ for

$f(x,y) = \sqrt{1-x^2}$ and $R = [-1,1] \times [-2,2]$.

Solution:



$n = 2$
$m = 4$
$mn = 8$ rectangles
use midpoints

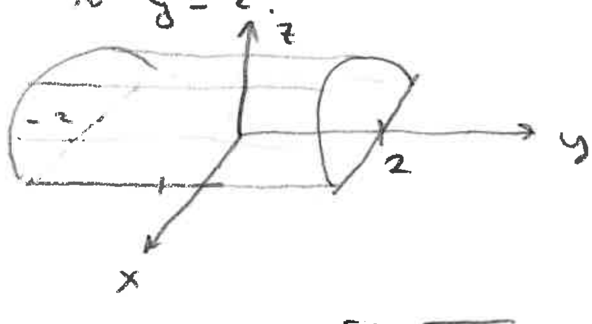
$$\iint_R \sqrt{1-x^2}\,dA \approx \underbrace{\sqrt{1-(\tfrac{1}{2})^2}}_{\text{height}}\underbrace{(1)}_{\text{area}}\underbrace{(4)}_{\text{rectangles}} + \sqrt{1-(-\tfrac{1}{2})^2}\,(1)\,4$$

$$= \frac{\sqrt{3}}{2}(4) + \frac{\sqrt{3}}{2}(4) = 4\sqrt{3} \approx 6.93$$

Example: Find $\iint_R f(x,y)\,dA$ for $f(x,y) = \sqrt{1-x^2}$ and $R = [-1,1] \times [-2,2]$ using volume of solid.

Solution: Let $z = f(x,y)$. Then $z = \sqrt{1-x^2} \Rightarrow x^2 + z^2 = 1$ a circle in the x-y plane, but $z \geq 0$.

Semicircle in x-y plane extending from $y=-2$ to $y=2$.



$$\text{volume} = \iint_R \sqrt{1-x^2}\,dA = \frac{\pi(1)^2}{2}(4) = 2\pi \approx 6.28$$