Fri 10/3

Test 2 Friday 10/10
Covers § 14.1 - 14.6
Review sheet of possible problems on Canvas

Quiz 7 Tuesday 10/14
§ 14.7
HW due Monday 10/20

§ 14.7 Maximum and Minimum Values

Together, these are called extrema or extreme values.

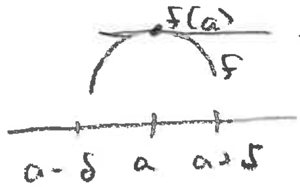
Let $f(\vec{x}) \in \mathbb{R}$ with domain D, and let $\vec{a} \in D$.

- $f(\vec{a})$ is a local (relative) maximum iff there exists $\delta > 0$ such that $\vec{x} \in D$ and $|\vec{x} - \vec{a}| < \delta$ implies $f(\vec{x}) \leq f(\vec{a})$.

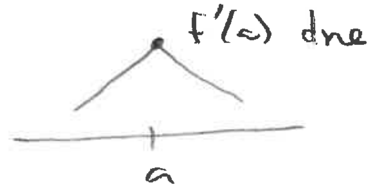
- $f(\vec{a})$ is a local (relative) minimum iff there exists $\delta > 0$ such that $\vec{x} \in D$ and $|\vec{x} - \vec{a}| < \delta$ implies $f(\vec{x}) \geq f(\vec{a})$.
- $f(\vec{a})$ is a global (absolute) maximum iff $\vec{x} \in D \Rightarrow f(\vec{x}) \leq f(\vec{a})$.
- $f(\vec{a})$ is a global (absolute) minimum iff $\vec{x} \in D \Rightarrow f(\vec{x}) \geq f(\vec{a})$.

Fri 10/3

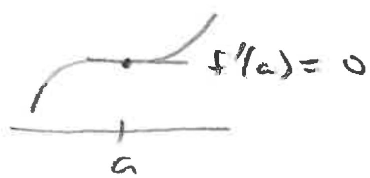
From Calculus I, Fermat's theorem said if $f(a)$ is a local extremum, then either $f'(a)=0$ or $f'(a)$ does not exist.



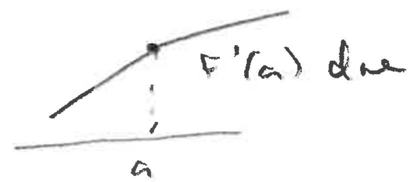
f(a) is a local max



This can be extend to $\mathbb{R}^2$ or $\mathbb{R}^3$. If $f(\vec{a})$ is an extremum, then either $\nabla f(\vec{a}) = 0$ or $\nabla f(\vec{a})$ dne.

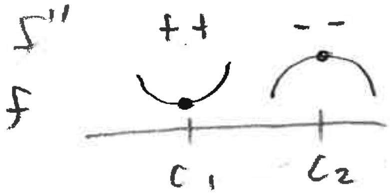


but f(a) is not a local max or local min



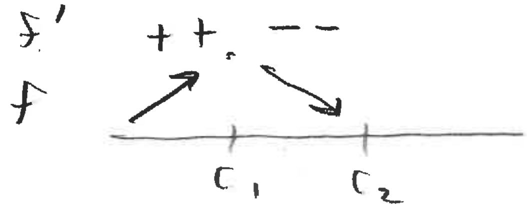
f(a) is not local extremum

A point $\vec{a}$ is a critical point (stationary point) iff either $\underbrace{\nabla f(\vec{a}) = 0}_{\text{we can test with 2nd derivative test}}$ or $\underbrace{\nabla f(\vec{a}) \text{ dne}}_{\text{we can't do anything with these, because we don't have 1st derivative test}}$.



$f'(c_1) = f'(c_2) = 0$

$f(c_1)$ is a local min
$f(c_2)$ is a local max



$(-\infty, c_1)$
(c_1, c_2)
(c_2, ∞)

$f(c_1)$ is local max

We can extend this to $\mathbb{R}^2$ and $\mathbb{R}^3$. We need to find $D^2_{\hat{u}} f$.

Let $f(x,y)$ such that $f_x(a,b) = 0$, $f_y(a,b) = 0$, i.e. (a,b) is a critical point, $f_{xy}(a,b) = f_{yx}(a,b)$.

$$D_{\hat{u}} f = \nabla f \cdot \hat{u} = \langle f_x, f_y \rangle \cdot \langle h, k \rangle$$
$$= h f_x + k f_y$$

where $\hat{u} = \langle h, k \rangle$.

$$D^2_{\hat{u}} f = D_{\hat{u}}(D_{\hat{u}} f) = h(D_{\hat{u}} f)_x + k(D_{\hat{u}} f)_y$$
$$= h(h f_x + k f_y)_x + k(h f_x + k f_y)_y$$
$$= h^2 f_{xx} + hk f_{yx} + kh f_{xy} + k^2 f_{yy}$$
$$= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$$
$$= f_{xx}\left(h^2 + 2\frac{f_{xy}}{f_{xx}} hk + \frac{f_{xy}^2}{f_{xx}^2} k^2\right) + k^2 f_{yy} - k^2 \frac{f_{xy}^2}{f_{xx}}$$
$$= f_{xx}\left(h + \frac{f_{xy}}{f_{xx}} k\right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx} f_{yy} - f_{xy}^2\right)$$

The only two that determine the sign of $D^2_{\hat{u}} f$ are f_{xx} and D, where

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

Case 1: $D > 0$ and $f_{xx} > 0$, $D^2 f > 0$, $f(a,b)$ is a local min

Case 2: $D > 0$ and $f_{xx} < 0$, $D^2 f < 0$, $f(a,b)$ is a local max.

Case 3: $D < 0$, then (a,b) is a saddle point.

When $D<0$, $D_{\vec{u}}^2 f$ changes sign. There is no local extremum at (a,b).

Example: Suppose $f_x(1,1) = f_y(1,1) = 0$.

(a) $f_{xx}(1,1)=4$, $f_{xy}(1,1)=1$, $f_{yy}(1,1)=2$

(b) $f_{xx}(1,1)=4$, $f_{xy}(1,1)=3$, $f_{yy}(1,1)=2$

Solution: (a) $D(1,1) = 4(2) - 1^2 = 7 > 0$

$f_{xx}(1,1) = 4 > 0$

so $f(1,1)$ is a local min.

(b) $D(1,1) = 4(2) - 3^2 = -1 < 0$

$f(1,1)$ is neither a local max nor local min.

$(1,1)$ is a saddle point