Wed 10/1

The derivative of f in the direction of $\hat{u}$ is

$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

Example: Let $f(x,y,z) = xy^2 \tan^{-1} z$ at the point $(2,1,1)$ in the direction $\vec{v} = \langle 1,1,1 \rangle$.

Solution: $\nabla f = \langle y^2 \tan^{-1} z, \ 2xy \tan^{-1} z, \ \dfrac{xy^2}{1+z^2} \rangle$

$$\nabla f(2,1,1) = \langle 1^2 \tan^{-1} 1, \ 2(2)1 \tan^{-1} 1, \ \frac{2(1)}{1+1^2} \rangle$$

$$= \langle \frac{\pi}{4}, \pi, 1 \rangle$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1,1,1 \rangle}{\sqrt{1^2+1^2+1^2}} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$$

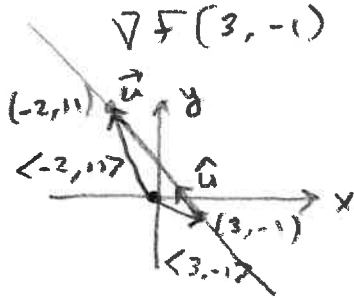
$$D_{\hat{v}} f = \langle \frac{\pi}{4}, \pi, 1 \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$$

$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{4} + \pi + 1 \right) = \frac{5\pi + 4}{4\sqrt{3}}$$

Example: Find the derivative of $f(x,y) = \dfrac{x}{y^2}$ at $(3,-1)$ in the direction of $(-2,11)$.

Solution: $\nabla f = \langle \dfrac{1}{y^2}, \dfrac{-2x}{y^3} \rangle$

$$\nabla f(3,-1) = \langle \frac{1}{(-1)^2}, \frac{-2(3)}{(-1)^3} \rangle = \langle 1, 6 \rangle$$



$$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$$

$$\langle 3,-1 \rangle + \vec{u} = \langle -2,11 \rangle \Rightarrow$$

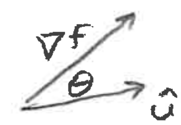
$$\vec{u} = \langle -2,11 \rangle - \langle 3,-1 \rangle = \langle -5,12 \rangle$$

$$\hat{u} = \frac{\langle -5,12 \rangle}{13}$$

$$D_{\hat{u}} f = \langle 1,6 \rangle \cdot \langle \frac{-5}{13}, \frac{12}{13} \rangle = \frac{67}{13}$$

Why is ∇f called the gradient?

$$D_{\hat{u}} f = \nabla f \cdot \hat{u} = |\nabla f|\,|\hat{u}| \cos\theta$$



$$= |\nabla f| \cos\theta$$

$$-1 \leq \cos\theta = \frac{D_{\hat{u}} f}{|\nabla f|} \leq 1 \Rightarrow -|\nabla f| \leq D_{\hat{u}} f \leq |\nabla f|$$

so $\theta = 0$, gives maximum rate of change of f, when $\hat{u}$ and ∇f are in the same direction.

Example: Find the maximum rate of change of $f(x,y,z) = x\ln(yz)$ at the point $(1, 2, \frac{1}{2})$, and find the direction in which it occurs.

Solution: $\nabla f = \langle \ln(yz), \frac{x}{y}, \frac{x}{z} \rangle$

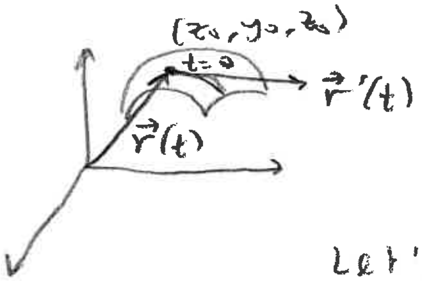
$$\nabla f(1, 2, \tfrac{1}{2}) = \langle \ln 1, \tfrac{1}{2}, 2 \rangle = \langle 0, \tfrac{1}{2}, 2 \rangle$$

$|\nabla f(1,2,\frac{1}{2})| = \sqrt{\frac{1}{4} + 4} = \frac{\sqrt{17}}{2}$ is the maximum rate of change of f, and it occurs in the direction $\langle 0, \frac{1}{2}, 2 \rangle$ or $\langle 0, 1, 4 \rangle$.

$$\hat{u} = \frac{\langle 0, 1, 4 \rangle}{\sqrt{17}}$$

$$D_{\hat{u}} f = \langle 0, \tfrac{1}{2}, 2 \rangle \cdot \frac{\langle 0, 1, 4 \rangle}{\sqrt{17}} = \frac{\frac{1}{2} + 8}{\sqrt{17}} = \frac{\frac{17}{2}}{\sqrt{17}} = \frac{\sqrt{17}}{2}$$

Eq. of a plane tangent to a point (x_0, y_0, z_0) on a level surface of $f(x,y,z)$.



$$f(x_0, y_0, z_0) = k$$

$$f(x,y,z) = k$$

Let's parametrize (x,y,z) to get a curve on the level surface.

$f(x(t), y(t), z(t))$, then using the chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0 \quad \Rightarrow$$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0 \quad \Rightarrow$$

$$\nabla f \cdot \vec{r}\,'(t) = 0$$

we know $\vec{r}\,'$ is tangent to the level surface.

we know that $\vec{r}\,'$ and ∇f are orthogonal.

Then ∇f is normal to a level surface.

So the eq. of the tangent plane is

$$\frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z} z = \frac{\partial f}{\partial x} x_0 + \frac{\partial f}{\partial y} y_0 + \frac{\partial f}{\partial z} z_0$$

Example: Find eq. of tangent plane of

$$xy + yz + xz = 5$$

at $(1,2,1)$, and find eq. of normal line.

Solution. Let $F(x,y,z) = xy + yz + xz = 5$ Level surface.

$\nabla F = \langle y+z, x+z, y+x \rangle$, $\nabla F(1,2,1) = \langle 3, 2, 3 \rangle$.

Eq. of tangent plane: $3x + 2y + 3z = 3(1) + 2(2) + 3(1) = 10$

Eq of normal line.

$$\left.\begin{array}{l} x = 1 + 3t \\ y = 2 + 2t \\ z = 1 + 3t \end{array}\right\} \Rightarrow \frac{x-1}{3} = \frac{y-2}{2} = \frac{z-1}{3}$$