Tues 9/30

§14.6 Directional Derivatives and Gradient Vector

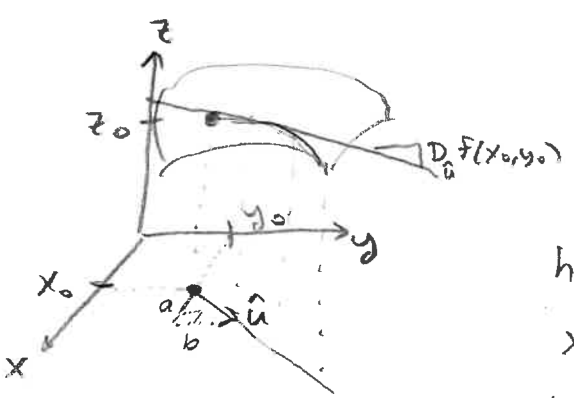
Test 2 Friday 10/10 covering §14.1 - 14.7.

So far for a multidimensional function $z = f(x,y)$, we have taken two derivatives, $\frac{\partial f}{\partial x}$ which is a derivative in the $\hat{\imath} = \langle 1, 0 \rangle$ and $\frac{\partial f}{\partial y}$ which is a derivative in the $\hat{\jmath} = \langle 0, 1 \rangle$. We can take a derivative in any direction $\hat{u} = \langle a, b \rangle$, where $|\hat{u}| = 1$ or $a^2 + b^2 = 1$. Recall in Calculus I, for $y = f(x)$,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Similarly, for $z = f(\vec{x})$, the directional derivative $D_{\hat{u}} f(\vec{x})$ is

$$D_{\hat{u}} f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h}$$



$z = f(x,y)$

$z_0 = f(x_0, y_0)$

$\hat{u} = \langle a, b \rangle$

$h\hat{u} = \langle ha, hb \rangle$

$x = x_0 + ha \Rightarrow \frac{dx}{dh} = a$

$y = y_0 + hb \Rightarrow \frac{dy}{dh} = b$

Under the special case where $\hat{u} = \hat{\imath}$ and $\hat{u} = \hat{\jmath}$,

$D_{\hat{\imath}} f = \frac{\partial f}{\partial x}$ and $D_{\hat{\jmath}} f = \frac{\partial f}{\partial y}$.

Since we have a parametrization, we can use the chain rule.

$$f(x,y) = f(x(h), y(h))$$

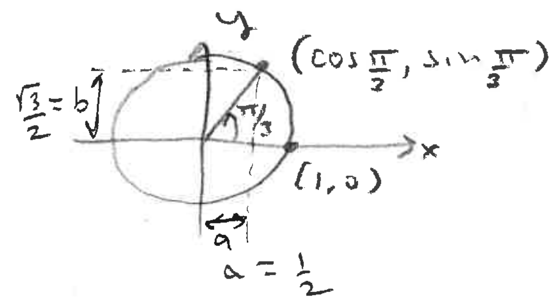
$$\frac{df}{dh} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle = D_{\hat{u}} f$$

$$D_{\hat{u}} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \hat{u}$$

Example: Let $f(x,y) = xy^3 - x^2$. Find the derivative of f at $(1,2)$ in the direction $\theta = \frac{\pi}{3}$.

Solution: $\hat{u} = \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle$



$$\hat{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy^3) - \frac{\partial}{\partial x}x^2 = y^3 - 2x$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = 2^3 - 2(1) = 6$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy^3) - \frac{\partial}{\partial y}x^2 = 3xy^2$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = 3(1)2^2 = 12$$

$$D_{\hat{u}} f(1,2) = \langle 6, 12 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$= 6\left(\frac{1}{2}\right) + 12\frac{\sqrt{3}}{2} = 3 + 6\sqrt{3}$$

The vector $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$ appears in the formula for $D_{\hat{u}} f$, we give it a name and a symbol. The name is gradient because it gives the grade (steepness of the hill) and the symbol ∇ called del. So

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

And in general,

$$D_{\hat{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \hat{u}$$

Example: Let $f(x,y,z) = x^2yz - xyz^3$, $P(2,-1,1)$, and $\hat{u} = \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle$.

Solution: $\nabla f = (2xyz - yz^3)\hat{\imath} + (x^2z - xz^3)\hat{\jmath} + (x^2y - 3xyz^2)\hat{k}$

$$\nabla f(2,-1,1) = (2(2)(-1)(1) - (-1)(1)^3)\hat{\imath} + (2^2(1) - 2(1)^3)\hat{\jmath} + (2^2(-1) - 3(2)(-1)(1)^2)\hat{k}$$

$$= \langle -3, 2, 2 \rangle$$

$$D_{\hat{u}} f(2,-1,1) = \langle -3,2,2 \rangle \cdot \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle$$

$$= 0 + \frac{8}{5} - \frac{6}{5} = \frac{2}{5}$$