Mon 9/29

§14.5 Chain Rule

- Implicit Differentiation
- Quiz 6 (Tuesday 9/30)

We take 10 of the 12 best quizzes in case you want to go to Engineering Expo.
You may also take quiz with another class.
1:35 (ANR 123) or 3:10 (MCS 202).

In calculus I, for $y = f(x)$, then we can differentiate explicity to get $y' = f'(x)$. Suppose we have $F(x,y) = C$, but with $y = f(x)$. Then we can differentiate implicitly using the chain rule, since

$$F(x,y) = F(x, y(x)) = C.$$

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\overbrace{\frac{dx}{dx}}^{1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \Rightarrow F_x + F_y\frac{dy}{dx} = 0 \Rightarrow$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

Example: For $\tan^{-1}(x^2y) = x + xy^2$, find $\frac{dy}{dx}$.

Solution: Let $F(x,y) = \tan^{-1}(x^2y) - x - xy^2 = 0.$

$$F_x = \frac{1}{1+(x^2y)^2}\frac{\partial}{\partial x}(x^2y) - 1 - y^2$$

$$= \frac{2xy}{1+x^4y^2} - 1 - y^2$$

$$= \frac{2xy - 1 - y^2 - x^4y^2 - x^4y^4}{1+x^4y^2}$$

$$F_y = \frac{x^2}{1+(x^2y)^2} - 2xy = \frac{x^2 - 2xy - 2x^5y^3}{1+x^4y^2}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{1+y^2+x^4y^2+x^4y^4-2xy}{x^2-2xy-2x^5y^3}$$

Alternatively,

$$\frac{d}{dx}\tan^{-1}(x^2y) = \frac{d}{dx}(x+xy^2) \Rightarrow$$

$$\frac{1}{1+x^4y^2}\frac{d}{dx}(x^2y) = 1 + \frac{d}{dx}(xy^2) \Rightarrow$$

$$\frac{2xy+x^2y'}{1+x^4y^2} = 1 + y^2 + 2xyy' \Rightarrow$$

$$2xy + x^2y' = 1 + y^2 + 2xyy' + x^4y^2 + x^4y^4 + 2x^5y^3y' \Rightarrow$$

$$x^2y' - 2xyy' - 2x^5y^3y' = 1 + y^2 + x^4y^2 + x^4y^4 - 2xy \Rightarrow$$

$$y' = \frac{1+y^2+x^4y^2+x^4y^4-2xy}{x^2-2xy-2x^5y^3}$$

Suppose $F(x,y,z) = C$, with $z = f(x,y)$.

Then $F(x,y,f(x,y)) = C$, and the chain rule gives

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial y}}_{0} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial y}}_{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

Example: For $x^2 - y^2 + z^2 - 2z = 4$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: $F(x,y,z) = x^2 - y^2 + z^2 - 2z = 4$

$F_x = 2x$

$F_y = -2y$

$F_z = 2z - 2$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-2x}{2z-2} = \frac{x}{1-z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{2y}{2z-2} = \frac{y}{z-1}$$

Alternatively,

$$\frac{\partial}{\partial x}(x^2-y^2+z^2-2z) = \frac{\partial 4}{\partial x} \Rightarrow 2x + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0 \Rightarrow$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2z-2} = \frac{x}{1-z}$$

$$\frac{\partial}{\partial y}(x^2-y^2+z^2-2z) = \frac{\partial 4}{\partial y} \Rightarrow -2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0 \Rightarrow$$

$$(2z-2)\frac{\partial z}{\partial y} = 2y \Rightarrow \frac{\partial z}{\partial y} = \frac{y}{z-1}$$

Example: For $e^z = xyz$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: $F(x,y,z) = e^z - xyz = 0$

$$F_x = -yz$$

$$F_y = -xz$$

$$F_z = e^z - xy$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{yz}{e^z - xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{xz}{e^z - xy}$$

Alternatively,

$$\frac{\partial}{\partial x} e^z = \frac{\partial}{\partial x}(xyz) \Rightarrow e^z \frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x} \Rightarrow$$

$$\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$$

$$\frac{\partial}{\partial y} e^z = \frac{\partial}{\partial y}(xyz) \Rightarrow e^z \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y} \Rightarrow$$

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$