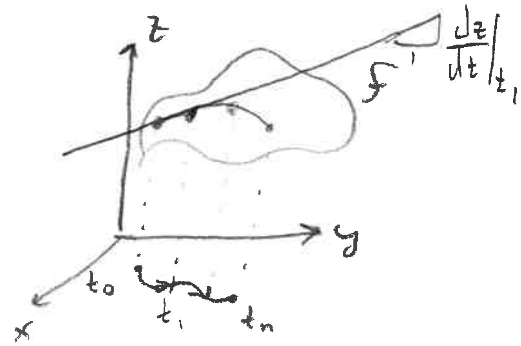
§14.5 The Chain Rule

Let $z = f(x,y)$. We parametrize x and y with $x = g(t)$ and $y = h(t)$. Then $z = f(g(t), h(t))$.

Using the total differential

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$



we have the derivative

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

which is the chain rule.

Example: $z = xy^3 - x^2y$, $x = t^2+1$, $y = t^2-1$.

Find $\frac{dz}{dt}$.

Solution: $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^3 - 2xy)2t + (3xy^2 - x^2)2t$$

Example: $w = f(x,y,z) = xe^{y/z}$, $x = t^2$, $y = 1-t$, $z = 1+2t$.

Find $\frac{dw}{dt}$.

Solution: $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= e^{y/z}2t + \frac{x}{z}e^{y/z}(-1) - \frac{xy}{z^2}e^{y/z}(2)$$

Example: Let $p(t) = f(x,y) = f(g(t), h(t))$, where f is differentiable, $g(2) = 4$, $g'(2) = -3$, $h(2) = 5$, $h'(2) = 6$, $f_x(4,5) = 2$, $f_y(4,5) = 8$. Find $p'(2)$.

Solution: $p'(t) = \dfrac{dp}{dt} = \dfrac{\partial p}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial p}{\partial y}\dfrac{dy}{dt}$

$$= f_x(g(t), h(t))\, g'(t) + f_y(g(t), h(t))\, h'(t)$$

$$p'(2) = f_x(g(2), h(2))\, g'(2) + f_y(g(2), h(2))\, h'(2)$$

$$= f_x(4,5)(-3) + f_y(4,5)\,6$$

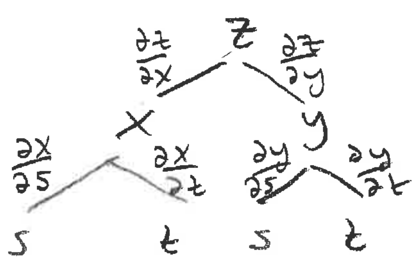
$$= 2(-3) + 8(6) = -6 + 48 = 42$$

Let $z = f(x,y) = f(g(s,t), h(s,t))$,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

when we do the partial derivative, we hold the other variables. So the same formula for ordinary derivative. We can use a tree diagram.



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Example: Let $z = e^r \cos\theta$, $r = st$, $\theta = \sqrt{s^2+t^2}$.

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution: $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$= (e^r \cos\theta)t - (e^r \sin\theta)\frac{s}{\sqrt{s^2+t^2}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$= (e^r \cos\theta)s - (e^r \sin\theta)\frac{t}{\sqrt{s^2+t^2}}$$

Example: $N = \frac{p+q}{p+r}$, $p = u + vw$, $q = v + uw$, $r = w + uv$.

Find $\frac{\partial N}{\partial u}$, $\frac{\partial N}{\partial v}$, $\frac{\partial N}{\partial w}$, when $(u,v,w) = (2,3,4)$.

Solution: $$\frac{\partial N}{\partial u} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial u} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial u} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial u}$$

$$= \frac{p+r-(p+q)}{(p+r)^2}(1) + \frac{1}{p+r}w - \frac{(p+q)}{(p+r)^2}v$$

$$= \frac{r-q+(p+r)w-(p+q)v}{(p+r)^2}$$

$p(2,3,4) = 2 + 3(4) = 14$

$q(2,3,4) = 3 + 2(4) = 11$

$r(2,3,4) = 4 + 2(3) = 10$

$$\frac{\partial N}{\partial u} = \frac{10-11+(14+10)4-(14+11)3}{(14+10)^2}$$

$$= \frac{-1+96-75}{24^2} = \frac{20}{24^2} = \frac{5}{12^2} = \frac{5}{144}$$