Wed 9/24

- Linearization
- Differentiability
- Total Differential

Recall the eq of a plane tangent to a function $z = f(x,y)$ at a point (x_0, y_0).

$$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The right-hand side, we call $L(x,y)$, which is a linearization of $f(x,y)$.

Example: Find the linearization $L(x,y)$ of the function $f(x,y) = 1 + x\ln(xy-5)$, at the point $(2,3)$.

Solution: $z_0 = f(2,3) = 1 + 2\ln(2(3)-5) = 1+2\ln 1 = 1$

$$f_x = \ln(xy-5) + \frac{xy}{xy-5}$$

$$f_x(2,3) = \ln(6-5) + \frac{6}{6-5} = 6$$

$$f_y = \frac{x^2}{xy-5}$$

$$f_y(2,3) = \frac{2^2}{6-5} = 4$$

$$L(x,y) = 1 + 6(x-2) + 4(y-3)$$
$$= 6x + 4y - 23$$

$L(2,3) = 12 + 12 - 23 = 1 = f(2,3)$

$L(2.1, 2.9) = 6(2.1) + 4(2.9) - 23 = 12.6 + 11.6 - 23 = 1.2$

$$f(2.1, 2.9) = 1 + 2.1 \ln(2.1(2.9) - 5)$$
$$\approx 1.181$$

$$\text{error} = \left| L(2.1, 2.9) - f(2.1, 2.9) \right| = 1.2 - 1.181 = 0.019$$

$$\Delta x = 2.1 - 2 = 0.1$$
$$\Delta y = 2.9 - 3 = -0.1$$

At $(2,3)$, error $= 0$. As $\Delta x, \Delta y \to 0$, the error $\to 0$.

Differentiability

We say a function $z = f(x,y)$ is differentiable at a point (x_0, y_0) iff the following equation holds.

$$\Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ when $(\Delta x, \Delta y) \to 0$.

It can be shown that f is differentiable at (x_0, y_0) if f_x and f_y are continuous at (x_0, y_0).

Example: Show that $f(x,y) = 4 \arctan(xy)$ is differentiable at $(1,1)$.

Solution: $f_x = \dfrac{4y}{1 + (xy)^2} = \dfrac{4y}{1 + x^2y^2}$

$f_y = \dfrac{4x}{1 + x^2y^2}$

Since $x^2y^2 \neq -1$, they are continuous on $\mathbb{R}^2$ and therefore at $(1,1)$.

$$L(x,y) = \pi + 2(x-1) + 2(y-1) = 2x + 2y + \pi - 4$$

Let $x = g(t)$ and $y = h(t)$, where $g'(t)$ and $h'(t)$ exists. Then $z = f(x,y) = f(g(t), h(t))$

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

Then take the limit as $\Delta t \to 0$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{dx}{dt}\lim_{\Delta t \to 0}\varepsilon_1 + \frac{dy}{dt}\lim_{\Delta t \to 0}\varepsilon_2$$

when $\Delta t \to 0$, then $(\Delta x, \Delta y) \to 0$, and $\varepsilon_1, \varepsilon_2 \to 0$.
we have the chain rule,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

If we leave off the dt, we have the total differential (differential)

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Example: Find the total differential of $z = e^{-2x}\cos 2\pi t$.

Solution: $dz = (-2e^{-2x}\cos 2\pi t)\,dx + (-2\pi e^{-2x}\sin 2\pi t)\,dt$

Example: If $z = 5x^2 + y^2$ and (x,y) changes from $(1,2)$ to $(1.05, 2.1)$, compare Δz and dz.

Solution:
$$\Delta z = f(1.05, 2.1) - f(1,2)$$
$$= 5(1.05)^2 + 2.1^2 - [5(1)^2 + 2^2] = 0.9225$$
$$dz = f_x(1,2)\,dx + f_y(1,2)\,dy$$
$$= 10(1)(1.05-1) + 2(2)(2.1-2)$$
$$= 0.5 + 0.4 = 0.9$$