Mon 9/22

Quiz 5 Tuesday 9/23

For $f(x,y) = 2x^2y^3 + 4xy^2$, find $\frac{\partial f}{\partial x}$ using the definition.

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④ $$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

② Substitution
$$= \lim_{h \to 0} \frac{[2(x+h)^2y^3 + 4(x+h)y^2] - (2x^2y^3 + 4xy^2)}{h}$$

② simplification
$$= \lim_{h \to 0} \frac{2x^2y^3 + 4xhy^3 + 2h^2y^3 + 4xy^2 + 4hy^2 - 2x^2y^3 - 4xy^2}{h}$$

$$= \lim_{h \to 0} \frac{4xhy^3 + 2h^2y^3 + 4hy^2}{h}$$

$$= \lim_{h \to 0} (4xy^3 + 2hy^3 + 4y^2)$$

① $$= 4xy^3 + 4y^2$$

Higher-Order Partial Derivatives

The second partial derivative of $z = f(x,y)$ with respect to x is indicated by

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

The mixed second partial derivatives are

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

↑ Do this first ↑ Do this first

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

Clairaut's theorem: If $f(x,y)$ exists on a disk D about (a,b), and f_{xy} and f_{yx} are continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Example: $f(x,y) = x^4y - 2x^3y^2$

$$f_x = 4x^3y - 6x^2y^2$$
$$f_y = x^4 - 4x^3y$$
$$f_{xx} = 12x^2y - 12xy^2$$
$$f_{yy} = \frac{\partial x^4}{\partial y} - 4\frac{\partial}{\partial y}(x^3y) = 0 - 4x^3\frac{\partial y}{\partial y} = -4x^3$$
$$f_{xy} = (4x^3y - 6x^2y^2)_y = 4x^3 - 12x^2y$$
$$f_{yx} = (x^4 - 4x^3y)_x = 4x^3 - 12x^2y$$

Note that $f_{xy} = f_{yx}$ for all $(x,y) \in \mathbb{R}^2$ because f exists on $\mathbb{R}^2$ and f_{xy} and f_{yx} are continuous on $\mathbb{R}^2$.

Example: Let $u = f(x,y,z) = x^a y^b z^c$, where $a,b,c \in \mathbb{R}$.

Find $\dfrac{\partial^6 u}{\partial x \partial y^2 \partial z^3} = f_{zzzyyx}$

Solution.
$$\frac{\partial u}{\partial z} = c\,x^a y^b z^{c-1}$$
$$\frac{\partial^2 u}{\partial z^2} = c(c-1)x^a y^b z^{c-2}$$
$$\frac{\partial^3 u}{\partial z^3} = c(c-1)(c-2)x^a y^b z^{c-3}$$
$$\frac{\partial^4 u}{\partial y \partial z^3} = bc(c-1)(c-2)x^a y^{b-1} z^{c-3}$$

$$\frac{\partial^5 u}{\partial y^2 \partial z^3} = b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}$$

$$\frac{\partial^6 u}{\partial x \partial y^2 \partial z^3} = ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}$$

Example: the ideal gas law is $PV = mRT$.

Show that $\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$

and $T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = mR$.

Solution: $\frac{\partial P}{\partial V} = \frac{\partial}{\partial V}\left(\frac{mRT}{V}\right) = -\frac{mRT}{V^2}$

$$\frac{\partial V}{\partial T} = \frac{\partial}{\partial T}\left(\frac{mRT}{P}\right) = \frac{mR}{P}$$

$$\frac{\partial T}{\partial P} = \frac{\partial}{\partial P}\left(\frac{PV}{mR}\right) = \frac{V}{mR}$$

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -\frac{mRT}{V^2}\frac{mR}{P}\frac{V}{mR} = -\frac{mRT}{VP} = -\frac{PV}{VP} = -1$$

$$T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = T\left(\frac{mR}{V}\right)\left(\frac{mR}{P}\right) = \frac{mRT}{PV}mR = mR$$

Example: Differentiate implicitly to find $\frac{\partial z}{\partial x}$ where $z = f(x,y)$ and

$$e^z = xyz.$$

Solution: $\frac{\partial e^z}{\partial x} = \frac{\partial}{\partial x}(xyz) \Rightarrow e^z\frac{\partial z}{\partial x} = y\frac{\partial}{\partial x}(xz) \Rightarrow$

$$e^z\frac{\partial z}{\partial x} = y\left(z\frac{\partial x}{\partial x} + x\frac{\partial z}{\partial x}\right) \Rightarrow$$

$$e^z\frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x} \Rightarrow$$

$$\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$$