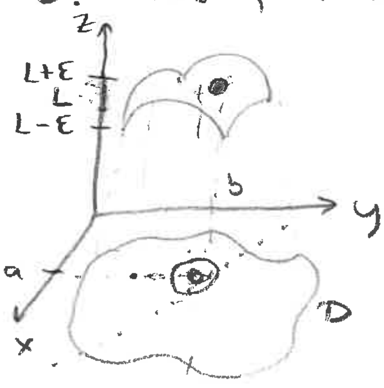
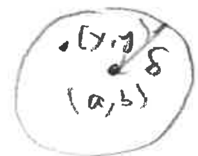
Wed 9/17

§14.2 Limits and Continuity

Let $\vec{x} = \langle x, y\rangle$ and $\vec{a} = \langle a, b\rangle$. Let $z = f(x,y)$ with domain D. The point (a,b) need not be in D.





$|\vec{x}-\vec{a}| < \delta$

$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\vec{x} \in D$ and $0 < |\vec{x}-\vec{a}| < \delta$ implies

$|f(\vec{x}) - L| < \varepsilon$.

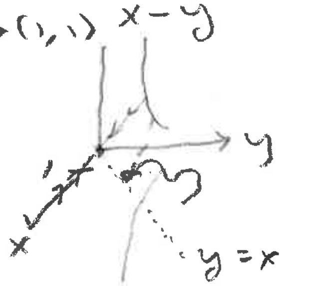
The function f is continuous at $\vec{x} = \vec{a}$ if and only if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$

Techniques for Finding Limits

1. Check for continuity (plug point in and see if it works.)

Example: $\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x-y) = \frac{\pi}{2}\sin(\pi - \frac{\pi}{2}) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}(1) = \frac{\pi}{2}$

Example: $\lim_{(x,y)\to(1,1)} \frac{3}{x-y}$ dne (infinite limit)



Example: $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2-y^2}{x-y}$

2. Rewrite the function

$$\lim_{(x,y)\to(1,1)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y)\to(1,1)} (x+y) = 1+1 = 2$$

Example: $$\lim_{(x,y)\to(1,-1)} \frac{x^2y+xy^2}{x^2-y^2} = \lim_{(x,y)\to(1,-1)} \frac{xy(x+y)}{(x+y)(x-y)}$$

$$= \lim_{(x,y)\to(1,-1)} \frac{xy}{x-y} = \frac{1(-1)}{1-(-1)} = -\frac{1}{2}$$

Example: $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}$$

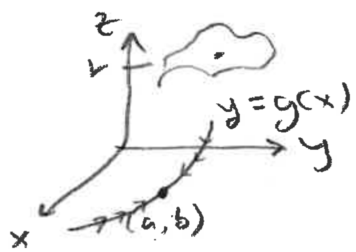
$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2+1-1}$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1 = 2$$

3. Check for jump discontinuities.

For $\lim\limits_{(x,y)\to(a,b)} f(x,y)$, if we have $\lim\limits_{(x,g(x))\to(a,b)} f(x,y) = L_1$

and $\lim\limits_{(x,h(x))\to(a,b)} f(x) = L_2 \neq L_1$, the $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ dne.



Example: $\lim_{(x,y)\to(0,0)} \dfrac{x^4-4y^2}{x^2+2y^2}$

Consider $\lim_{(x,0)\to(0,0)} \dfrac{x^4-4y^2}{x^2+2y^2} = \lim_{x\to 0} \dfrac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0$

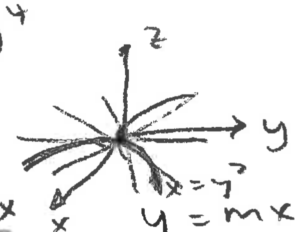
Consider $\lim_{(0,y)\to(0,0)} \dfrac{x^4-4y^2}{x^2+2y^2} = \lim_{y\to 0} \dfrac{-4y^2}{2y^2} = -2 \neq 0$

The limit dne.

Example: $\lim_{(x,y)\to(0,0)} \dfrac{xy^2\cos y}{3x^2+y^4}$

Consider $\lim_{(x,0)\to(0,0)} \dfrac{xy^2\cos y}{3x^2+y^4} = \lim_{x\to 0} \dfrac{0}{3x^2} = 0$

Consider $\lim_{(0,y)\to(0,0)} \dfrac{xy^2\cos y}{3x^2+y^4} = \lim_{y\to 0} \dfrac{0}{y^4} = 0$



Take $y = mx$

$\lim_{(x,mx)\to(0,0)} \dfrac{xy^2\cos y}{3x^2+y^4} = \lim_{x\to 0} \dfrac{m^2x^3\cos mx}{3x^2+m^4x^4}$

$= \lim_{x\to 0} \dfrac{m^2x\cos mx}{3+m^4x^2} = \dfrac{m^2(0)(1)}{3+0} = \dfrac{0}{3} = 0$

Let $x = y^2$.

$\lim_{(y^2,y)\to(0,0)} \dfrac{xy^2\cos y}{3x^2+y^4} = \lim_{y\to 0} \dfrac{y^2y^2\cos y}{3(y^2)^2+y^4}$

$= \lim_{y\to 0} \dfrac{y^4\cos y}{3y^4+y^4} = \lim_{y\to 0} \dfrac{y^4\cos y}{4y^4}$

$= \lim_{y\to 0} \dfrac{\cos y}{4} = \dfrac{1}{4} \neq 0$

Limit dne