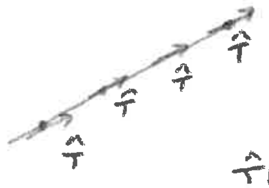
Quiz 3 9/9 at 4:50.

Fri 9/12 Test 1

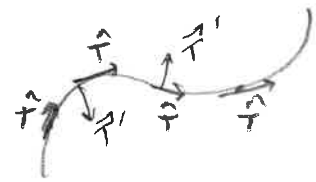
Wed 9/10 Work Review Problems on Canvas

Tues 9/9 Finish § 13.3 Curvature HW 13.3 due 9/19.

Curvature



A line has no curvature.
$\hat{T}(t)$ does not change.



This has curvature

$|\hat{T}(t)| = 1$

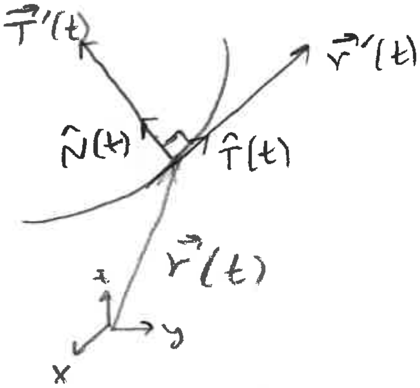
Curvature is related to the derivative of $\hat{T}(t)$.
Three things to note about the derivative are:

(i) $\vec{T}'(t)$ is not a unit vector.

(ii) Since $|\hat{T}(t)| = 1$ is a constant, then

$$\hat{T}(t) \cdot \vec{T}'(t) = 0$$

(iii) $\vec{T}'(t)$ points toward the inside or concave side of the curve.



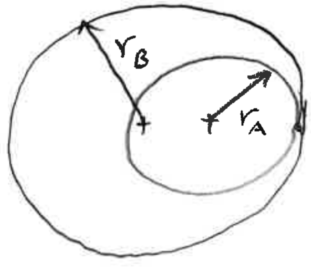
$$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

we define the unit normal vector as

$$\hat{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

The TNB frame.

Consider two circles with radii r_A and r_B.



Obviously circle A has more curvature than circle B, but $r_A < r_B$.
So curvature is inversely proportional to the radius of a circle.

In HW 13.3, you will show that curvature is one over radius for a circle. We give curvature the symbol Kappa κ. So for a circle of radius r,

$$\kappa = \frac{1}{r}$$

which means curvature has units of ft^{-1} or m^{-1}.
So we take derivative of $\hat{T}(t)$ with respect to s.

$$\kappa = \left| \frac{d\hat{T}(t)}{ds} \right| = \left| \frac{d\hat{T}(t)}{dt} \frac{dt}{ds} \right| = \left| \frac{d\hat{T}(t)/dt}{ds/dt} \right|$$

$$= \left| \frac{\vec{T}'(t)}{|\vec{r}'(t)|} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

Example: Let $\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle$.
Find unit tangent and unit normal vectors, and find curvature.

Solution: $\vec{r}'(t) = \langle 1, -3\sin t, 3\cos t \rangle$

$$|\vec{r}'(t)| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$$

$$\hat{T}(t) = \frac{\langle 1, -3\sin t, 3\cos t \rangle}{\sqrt{10}} = \left\langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\sin t, \frac{3}{\sqrt{10}}\cos t \right\rangle$$

$$\vec{T}'(t) = \left\langle 0, \frac{-3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle$$

$$|\vec{T}'(t)| = \sqrt{0^2 + \frac{9}{10}\cos^2 t + \frac{9}{10}\sin^2 t} = \frac{3}{\sqrt{10}}$$

$$\hat{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle 0, -\cos t, -\sin t \rangle$$

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$$

Since κ depends on $\vec{T}'(t)$ and $\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$,

we see that κ depends on $\vec{r}''(t)$, which is acceleration.

$$\vec{r}'(t) = |\vec{r}'(t)|\,\hat{T}(t) = \frac{ds}{dt}\hat{T}(t)$$

$$\vec{r}''(t) = \frac{d^2s}{dt^2}\hat{T}(t) + \frac{ds}{dt}\vec{T}'(t)$$

$$\vec{r}'(t) \times \vec{r}''(t) = \frac{ds}{dt}\frac{d^2s}{dt^2}\overbrace{\left(\hat{T}(t) \times \hat{T}(t)\right)}^{0} + \left(\frac{ds}{dt}\right)^2\left(\hat{T}(t) \times \vec{T}'(t)\right)$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = |\vec{r}'(t)|^2\,|\hat{T}(t)|\,|\vec{T}'(t)| \sin\frac{\pi}{2}$$

$$= |\vec{r}'(t)|^2\,|\vec{T}'(t)| \Rightarrow$$

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Example: Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1,1,1)$.

Solution: $\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}''(t) = \langle 0, 2, 6t \rangle$

$\vec{r}(t) = \langle 1, 1, 1 \rangle \Rightarrow t = 1$

$\vec{r}'(1) = \langle 1, 2, 3 \rangle$

$\vec{r}''(1) = \langle 0, 2, 6 \rangle$

$$\vec{r}'(1) \times \vec{r}''(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\hat{i} - 6\hat{j} + 2\hat{k}$$

$$|\vec{r}\,'(1) \times \vec{r}\,''(1)| = \sqrt{36+36+4}$$

$$k = \frac{\sqrt{76}}{|\langle 1,2,3\rangle|^3} = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{2\sqrt{19}}{14\sqrt{14}} = \frac{1}{7}\sqrt{\frac{19}{14}}$$

TNB frame: $\hat{B} = \hat{T} \times \hat{N}$ called binormal unit vector.