Fri 10/5

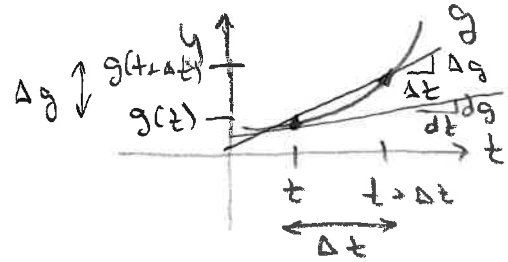
Test I Friday 9/12 covers sections 12.1-12.5 and 13.1-13.2. Review sheets on Canvas.

§13.2 Derivatives and Integrals of Vector Functions

Derivatives

Recall from Calculus I that for $y = g(t)$, the derivative is defined as

$$\frac{dg}{dt} = \lim_{\Delta t \to 0} \frac{\Delta g}{\Delta t} = \frac{g(t+\Delta t) - g(t)}{t+\Delta t - t} = g'(t)$$



Now consider $\mathbb{R}^3$ and the function

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

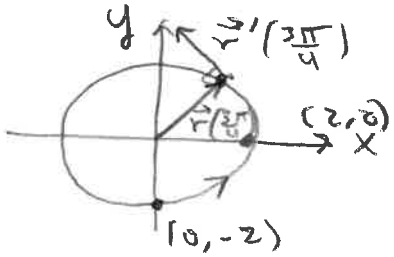
$$= \lim_{\Delta t \to 0} \frac{\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle - \langle f(t), g(t), h(t)\rangle}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

Example: $\vec{r}(t) = 2\sin t\,\hat{i} - 2\cos t\,\hat{j}$



$\vec{r}(0) = \langle 0, -2 \rangle$

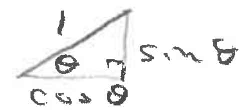
$\vec{r}(\frac{\pi}{2}) = \langle 2, 0 \rangle$

$x = 2\sin t \Rightarrow x^2 = 4\sin^2 t$

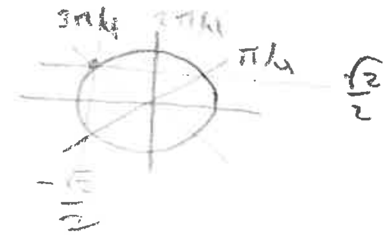
$y = -2\cos t \Rightarrow y^2 = 4\cos^2 t$

$x^2 + y^2 = 4\sin^2 t + 4\cos^2 t = 4(\sin^2 t + \cos^2 t)$

$x^2 + y^2 = 4$

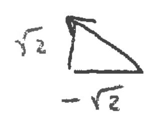


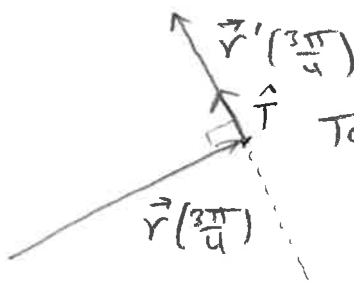
$\vec{r}(\frac{3\pi}{4}) = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$



$\vec{r}'(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$

$\vec{r}'(\frac{3\pi}{4}) = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$





Tangent unit vector

$$\hat{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\hat{T}(\tfrac{3\pi}{4}) = \frac{\vec{r}'(\frac{3\pi}{4})}{|\vec{r}'(\frac{3\pi}{4})|} = \frac{\langle -\sqrt{2}, \sqrt{2} \rangle}{\sqrt{2+2}} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

The eq of the tangent line at $t = \frac{3\pi}{4}$ is

$x = \sqrt{2} - \sqrt{2}\,t$ $\qquad$ $x = \vec{r}(\frac{3\pi}{4})\cdot\hat{i} + (\vec{r}'(\frac{3\pi}{4})\cdot\hat{i})\,t$

$y = \sqrt{2} + \sqrt{2}\,t$

Derivative Properties

Sum Rule: $\frac{d}{dt}(\vec{u}(t)+\vec{v}(t)) = u'(t) + \vec{v}'(t)$

Product Rule: $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

special case, $f(t) = c$.

$$\frac{d}{dt}(c\,\vec{u}(t)) = c\,\vec{u}'(t)$$

$$\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

$$\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$$

Chain Rule: $\frac{d}{dt}\vec{u}(f(t)) = \vec{u}'(f(t))\, f'(t)$

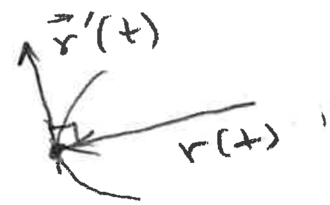
If $|\vec{r}(t)| = c$, then

$$\vec{r}(t)\cdot\vec{r}'(t) = \frac{1}{2}\, 2\,\vec{r}(t)\cdot\vec{r}'(t)$$

$$= \frac{1}{2}(\vec{r}(t)\cdot\vec{r}'(t) + \vec{r}'(t)\cdot\vec{r}(t))$$

$$= \frac{1}{2}\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] = \frac{1}{2}\frac{d}{dt}|\vec{r}|^2 = \frac{1}{2}\frac{d}{dt}c^2$$

$$= 0$$



It can be shown that,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \vec{r}(t_i)\Delta t = \int_a^b \vec{r}(t)\,dt$$

$$= \left< \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right>$$

Example: $\int_0^1 \left(\frac{1}{t+1}\hat{i} + \frac{1}{t^2+1}\hat{j} + \frac{t}{t^2+1}\hat{k} \right) dt$

$$= \int_0^1 \frac{1}{t+1}\,dt\,\hat{i} + \int_0^1 \frac{1}{t^2+1}\,dt\,\hat{j} + \int_0^1 \frac{t}{t^2+1}\,dt\,\hat{k}$$

$$= \ln(t+1)\Big|_0^1 \hat{i} + \tan^{-1} t\Big|_0^1 \hat{j} + \frac{1}{2}\int_0^1 \frac{1}{t^2+1}\,d(t^2+1)\,\hat{k}$$

$$= \ln 2\,\hat{i} + \frac{\pi}{4}\hat{j} + \frac{1}{2}\ln(t^2+1)\Big|_0^1 \hat{k}$$

$$= \ln 2\,\hat{i} + \frac{\pi}{4}\hat{j} + \frac{1}{2}\ln 2\,\hat{k}$$

Example: $\vec{r}\,'(t) = t\,\hat{i} + e^t\hat{j} + te^t\hat{k}$

$\vec{r}(0) = \hat{i} + \hat{j} + \hat{k}$

Find $\vec{r}(t)$.

Solution: $\vec{r}(t) = \int \vec{r}\,'(t)\,dt$

$$= \frac{t^2}{2}\hat{i} + e^t\hat{j} + \int te^t\,dt\,\hat{k} + \vec{c}$$

$$= \frac{t^2}{2}\hat{i} + e^t\hat{j} + \int t\,de^t\,\hat{k} + \vec{c}$$

$$= \frac{t^2}{2}\hat{i} + e^t\hat{j} + \left(te^t - \int e^t\,dt\right) + \vec{c}$$

$$= \frac{t^2}{2}\hat{i} + e^t\hat{j} + (te^t - e^t)\hat{k} + \vec{c}$$

$\vec{r}(0) = \hat{j} - \hat{k} + \vec{c} = \hat{i} + \hat{j} + \hat{k} \Rightarrow$

$\vec{c} = \hat{j} + 2\hat{k}$