Fri 8/29

§12.5 Equations of Lines and Planes

Recall that in $\mathbb{R}^1$, we need two pieces of information to write the eq. of a line:

- point and a slope
- two points

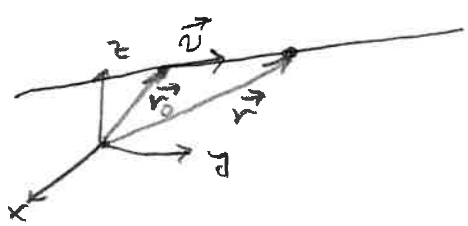
Case 1: Given a point (x_0, y_0) and a slope m

$$m = \frac{y - y_0}{x - x_0} \Rightarrow y = m(x - x_0) + y_0$$

Let $t = x - x_0$. Then the eq becomes

$$y = mt + y_0$$

Extending this to $\mathbb{R}^3$, we have the following:



Let $\vec{v} = \langle a, b, c \rangle$

$$\vec{r} = \langle x, y, z \rangle$$
$$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$$

$\vec{r} = \vec{r_0} + \vec{v}t$ This is the vector eq. of the line.

$$\vec{r} = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t$$
$$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \text{parametric eqs.}$$

$$\underbrace{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}_{\text{symmetric eqr.}}$$

This is Quiz #2 on Tuesday 9/2.

Example: Write the vector eq., parametric eqs, and symmetric eqs. for a line passing through $(5,1,3)$ and parallel to $\langle 1,4,-2\rangle$.

Solution: ① Name

(3) $\vec{r} = \vec{r}_0 + \vec{v}t = \langle 5,1,3\rangle + \langle 1,4,-2\rangle t$

$= \langle 5+t,\ 1+4t,\ 3-2t\rangle$

(3)
$$x = 5+t$$
$$y = 1+4t$$
$$z = 3-2t$$

(3) $x-5 = \frac{y-1}{4} = \frac{z-3}{-2}$

Case 2: Given two points (x_0,y_0) and (x_1,y_1).

$$\frac{y-y_0}{x-x_0} = \frac{y_1-y_0}{x_1-x_0} \Rightarrow y = (y_1-y_0)\left(\frac{x-x_0}{x_1-x_0}\right) + y_0$$

Let $t = \frac{x-x_0}{x_1-x_0}$. Then the eq. becomes

$$y = (y_1-y_0)t + y_0 = (1-t)y_0 + ty_1$$

In $\mathbb{R}^3$, the eq. of a line passing through $\vec{r}_0$ and $\vec{r}_1$ is

$$\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1 .$$

Example: Find the vector eq, parametric eq, and symmetric eqs. for a line passing through $(-8,1,4)$ and $(3,-2,5)$.

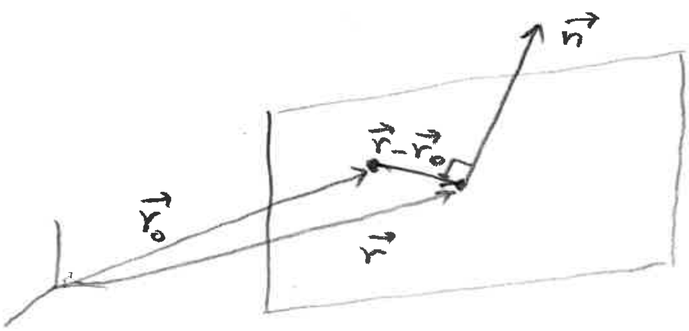
Solution:
$$\begin{aligned}\vec{r} &= (1-t)\vec{r}_0 + t\vec{r}_1 \\ &= (1-t)\langle -8,1,4\rangle + t\langle 3,-2,5\rangle \\ &= \langle -8+8t,\ 1-t,\ 4-4t\rangle + \langle 3t,-2t,5t\rangle \\ &= \langle -8+11t,\ 1-3t,\ 4+t\rangle\end{aligned}$$

$$x = -8+11t \Rightarrow t = \frac{x+8}{11}$$
$$y = 1-3t \Rightarrow t = \frac{y-1}{-3}$$
$$z = 4+t \Rightarrow t = z-4$$

$$\frac{x+8}{11} = \frac{y-1}{-3} = z-4$$

Equation of a plane

To write the eq. we need a point (x_0, y_0, z_0) and a vector that is orthogonal to every vector in the plane, called a normal vector $\vec{n} = \langle a,b,c\rangle$.



$$\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0 \Rightarrow$$
$$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0 \Rightarrow$$
$$ax+by+cz = \underbrace{ax_0+by_0+cz_0}_{-d}$$

$$ax+by+cz+d = 0$$

Example: Find the eq. of a plane containing $(1,3,-1)$ and perpendicular to the line

$$\frac{x+3}{4} = -y = \frac{z-1}{5}.$$

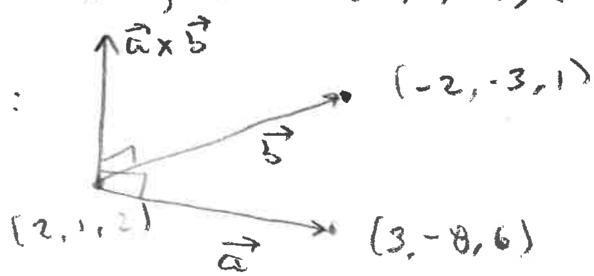
Solution: $\vec{n} = \langle 4, -1, 5 \rangle$

$$4x - y + 5z = 4(1) - 3 + 5(-1) = -4$$

$$4x - y + 5z = -4$$

Example: Find the eq. of a plane containing the points $(2,1,2)$, $(3,-8,6)$, and $(-2,-3,1)$.

Solution:



$\vec{a} = \langle 1, -9, 4 \rangle$

$\vec{b} = \langle -4, -4, -1 \rangle$

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -9 & 4 \\ -4 & -4 & -1 \end{vmatrix} = \hat{i}(9+16) - \hat{j}(-1+16) + \hat{k}(-4-36)$$

$$= 25\hat{i} - 15\hat{j} - 40\hat{k}$$

$$25x - 15y - 40z = 25(2) - 15(1) - 40(2) = -45 \Rightarrow$$

$$5x - 3y - 8z = -9$$