Wed 8/27

§12.4 Cross Product

$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

$\langle 2,-1,1\rangle \times \langle 3,2,-4\rangle = \langle 2,11,7\rangle$

<u>Determinants</u>

The determinant of a 2×2 (two rows and 2 columns) matrix is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{1+1} a_{11}a_{22} + (-1)^{1+2} a_{12}a_{21}$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Similarly, the determinant of a 3×3 matrix is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23})$$
$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

To find the cross product with determinants, we write

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$\langle 2,-1,1\rangle \times \langle 3,2,-4\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 1 \\ 2 & -4 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= \hat{i}(4-2) - \hat{j}(-8-3) + \hat{k}(4+3)$$

$$= 2\hat{i} + 11\hat{j} + 7\hat{k}$$

$$\langle 2,-1,1\rangle \times \langle 3,2,-4\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix}$$

$$= -\hat{j}\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} + (-1)\begin{vmatrix} \hat{i} & \hat{k} \\ 3 & -4 \end{vmatrix} - 2\begin{vmatrix} \hat{i} & \hat{k} \\ 2 & 1 \end{vmatrix}$$

$$= -\hat{j}(-8-3) - 1(-4\hat{i} - 3\hat{k}) - 2(\hat{i} - 2\hat{k})$$

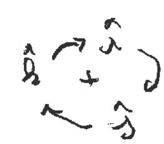
$$= 11\hat{j} + 4\hat{i} + 3\hat{k} - 2\hat{i} + 4\hat{k} = 2\hat{i} + 11\hat{j} + 7\hat{k}$$

$$(2\hat{i} - \hat{j} + \hat{k}) \times (3\hat{i} + 2\hat{j} - 4\hat{k})$$

$$= 6\,\hat{i}\times\hat{i} + 4\,\hat{i}\times\hat{j} - 8\,\hat{i}\times\hat{k}$$
$$-3\,\hat{j}\times\hat{i} - 2\,\hat{j}\times\hat{j} + 4\,\hat{j}\times\hat{k}$$
$$+3\,\hat{k}\times\hat{i} + 2\,\hat{k}\times\hat{j} - 4\,\hat{k}\times\hat{k}$$

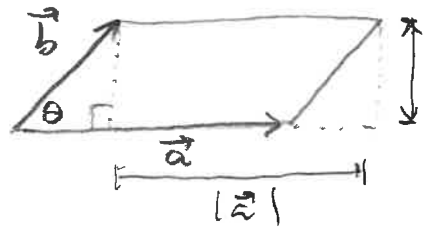
$$= 4\hat{k} + 8\hat{j} + 3\hat{k} + 4\hat{i} + 3\hat{j} - 2\hat{i}$$

$$= 2\hat{i} + 11\hat{j} + 7\hat{k}$$



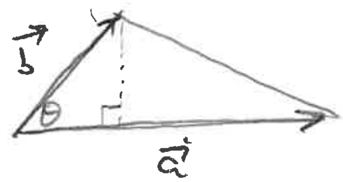
$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$
$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Consider a parallelogram



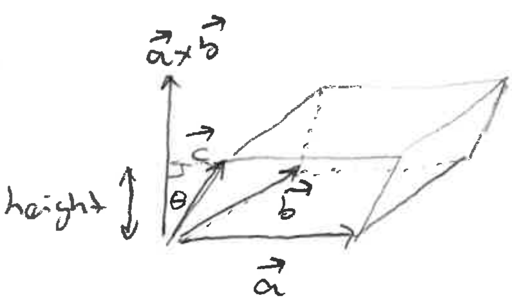
$$\text{Area} = |\vec{a}||\vec{b}|\sin\theta = |\vec{a}\times\vec{b}|$$

Consider a triangle



$$\text{Area} = \tfrac{1}{2}(\text{base})(\text{height})$$
$$= \tfrac{1}{2}|\vec{a}||\vec{b}|\sin\theta$$
$$= \tfrac{1}{2}|\vec{a}\times\vec{b}|$$

Consider a parallelepiped.



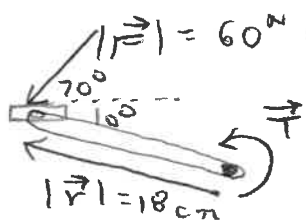
$$\text{Volume} = \text{area} * \text{height}$$
$$= |\vec{a}\times\vec{b}|\,|\vec{c}|\cos\theta$$
$$= (\vec{a}\times\vec{b})\cdot\vec{c}$$

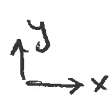
This is the scalar triple product.

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{c}\cdot(\vec{a}\times\vec{b}) = (\vec{c}\times\vec{a})\cdot\vec{b} = (\vec{b}\times\vec{c})\cdot\vec{a}$$

Torque: $\vec{T} = \vec{r}\times\vec{F}$

Example:





$$\vec{r} = -18\cos 10^\circ\,\hat{i} + 18\sin 10^\circ\,\hat{j}$$
$$\vec{F} = -60\cos 70^\circ\,\hat{i} - 60\sin 70^\circ\,\hat{j}$$

$$\vec{T} = \vec{r}\times\vec{F}$$
$$= 1080\cos 10^\circ\sin 70^\circ + 1080\sin 10^\circ\cos 70^\circ$$
$$= 1080\sin 80^\circ\ \text{N}\cdot\cancel{\text{cm}}\left(\frac{1\,\text{m}}{100\,\cancel{\text{cm}}}\right) \approx 10.64\ \text{N}\cdot\text{m}$$