Mon 8/25

§ 2.3 Dot Product

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ be two vectors. The dot product is

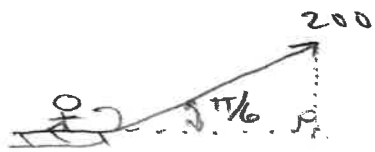
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Quiz #1 Tuesday 4:50

Sled Problem

A sled is pulled a distance 40 ft along a horizontal line by a force of 200 lb at an angle of $\pi/6$ with the horizontal. Calculate the work.

(1) Name



(2) $\vec{F} = 200 \cos\frac{\pi}{6}\,\hat{\imath} + 200 \sin\frac{\pi}{6}\,\hat{\jmath}$

(2) $\vec{D} = 40\,\hat{\imath}$

(2) $W = \vec{F} \cdot \vec{D} = \langle 100\sqrt{3}, 100 \rangle \cdot \langle 40, 0 \rangle$ (1)

$= 4000\sqrt{3}$ ft·lb
(1) (1)

Properties of Dot Product

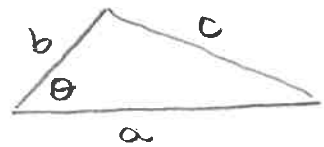
(i) $\vec{a} \cdot \vec{a} = |\vec{a}|^2 \geq 0$ positivity

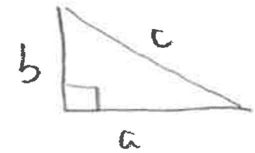
(ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ commutativity

(iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

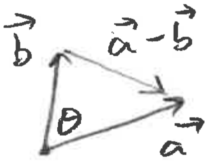
(iv) $(c\vec{a}) \cdot \vec{b} = \vec{a} \cdot (c\vec{b}) = c(\vec{a} \cdot \vec{b})$

Law of Cosines





$a^2 + b^2 - 2ab\cos\theta = c^2$ $\qquad$ $a^2 + b^2 = c^2$



$$|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta = |\vec{a} - \vec{b}|^2$$

$$= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

$$= (\vec{a} - \vec{b}) \cdot \vec{a} + (\vec{a} - \vec{b}) \cdot (-\vec{b})$$

$$= \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

$$= |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \quad \Rightarrow$$

$$-2|\vec{a}||\vec{b}|\cos\theta = -2\vec{a} \cdot \vec{b} \quad \Rightarrow$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

If $\vec{a}$ and $\vec{b}$ are orthogonal

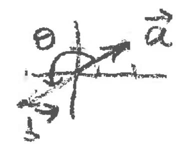
$$\vec{a} \cdot \vec{b} = 0$$

If $\vec{a}$ and $\vec{b}$ are parallel, then there exists $c \in \mathbb{R}$ such that $\vec{a} = c\vec{b}$.

$\theta = 0$

$$\vec{a} \cdot \vec{b} = c\vec{b} \cdot \vec{b} = c|\vec{b}|^2 = |\vec{a}||\vec{b}|$$

Example: $\vec{a} = \langle 2, 1 \rangle$, $\vec{b} = \langle -2, -1 \rangle$



$\vec{a} \cdot \vec{b} = -4 - 1 = -5$

$|\vec{a}| = \sqrt{5}$ $\qquad$ $|\vec{a}||\vec{b}|\cos\theta = \sqrt{5}\sqrt{5}(-1) = -5$

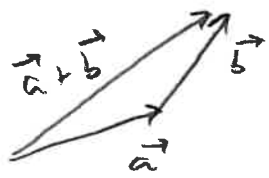
$|\vec{b}| = \sqrt{5}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \hat{a}\cdot\hat{b} \qquad \hat{a} = \frac{\vec{a}}{|\vec{a}|} \qquad |\hat{a}| = 1$$

$$-1 \le \cos\theta \le 1 \Rightarrow |\cos\theta| \le 1$$

$$\left|\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right| \le 1 \Rightarrow |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$$

Cauchy-Schwarz Inequality



$$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$$

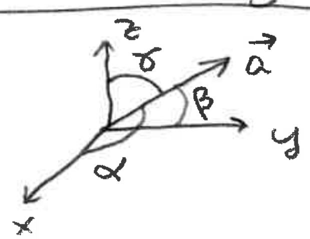
Triangle Inequality

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \le |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2$$

$$= (|\vec{a}| + |\vec{b}|)^2 \Rightarrow |\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$$

Direction Angles



$$\cos\alpha = \frac{a_1}{|\vec{a}|}$$

$$\cos\beta = \frac{a_2}{|\vec{a}|}$$

$$\cos\gamma = \frac{a_3}{|\vec{a}|}$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{(|\vec{a}|\cos\alpha)^2 + (|\vec{a}|\cos\beta)^2 + (|\vec{a}|\cos\gamma)^2}$$

$$= |\vec{a}|\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Let $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{4}$. Find γ.

$$\cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{4} + \cos^2\gamma = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \cos^2\gamma = 1 \Rightarrow \cos^2\gamma = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$$

$$\Rightarrow \cos\gamma = \pm\frac{1}{2} \Rightarrow \gamma = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$