Example: show that the following eq. is an eq. of a sphere.

$$x^2+y^2+z^2+4x-6y-2z=22$$

$$x^2+4x+4+y^2-6y+9+z^2-2z+1=22+4+9$$

$$(x+2)^2+(y-3)^2+(z-1)^2=36$$

center is $(-2,3,1)$

radius is 6

Example: Find an eq of a sphere with points $P(x,y,z)$ such that the distance from P to $A(-3,6,2)$ is twice the distance from P to $B(6,2,-1)$.

$$|PA| = 2|PB| \Rightarrow |PA|^2 = 4|PB|^2 \Rightarrow$$

$$(x+3)^2+(y-6)^2+(z-2)^2 = 4[(x-6)^2+(y-2)^2+(z+1)^2]$$

$$x^2+6x+9+y^2-12y+36+z^2-4z+4$$

$$=4(x^2-12x+36+y^2-4y+4+z^2+2z+1)$$

$$=4x^2-48x+144+4y^2-16y+16+4z^2+8z+4 \Rightarrow$$

$$3x^2-54x+3y^2-4y+3z^2+12z=9+36+4-144-16-4$$

$$=-115 \Rightarrow$$

$$x^2-18x+81+y^2-\frac{4}{3}y+\frac{4}{9}+z^2+4z+4=-\frac{115}{3}+81+\frac{4}{9}$$

$$(x-9)^2+(y-\tfrac{2}{3})^2+(z+2)^2=\frac{424}{9}$$

center is $(9,\frac{2}{3},-2)$

radius is $\frac{\sqrt{424}}{3}$

Wed 8/20

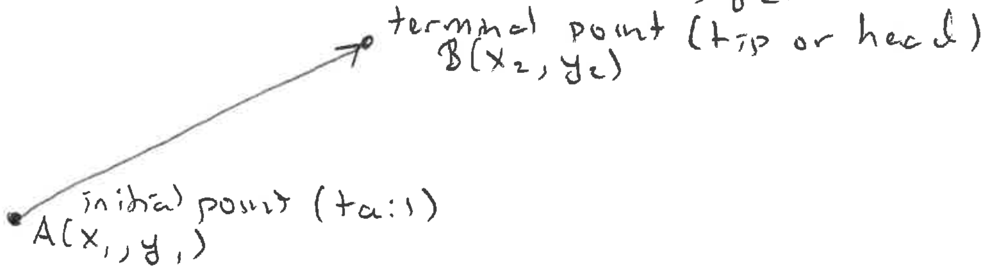
§12.2 Vectors

Definition

A vector is quantity that has magnitude and direction. We indicate a vector with the notation

$$\vec{v}.$$

We can construct a vector from a directed line segment. Consider $A(x_1, y_1)$ and $B(x_2, y_2)$.



$$\vec{v} = \overrightarrow{AB}$$

We indicate the length of vector $\vec{v}$ with $|\vec{v}|$.

We calculate this length as

$$|\vec{v}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

The coordinates of A are (x_1, y_1).

The components of $\vec{v}$ are $\langle v_1, v_2 \rangle$.

So in this case,

$$v_1 = x_2 - x_1$$
$$v_2 = y_2 - y_1$$

In general, the length of a vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$ is

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$$= 4(x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 2z + 1)$$

$$= 4x^2 - 48x + 144 + 4y^2 - 16y + 16 + 4z^2 + 8z + 4 \Rightarrow$$

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$$= -115 \Rightarrow$$

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Wed 8/20

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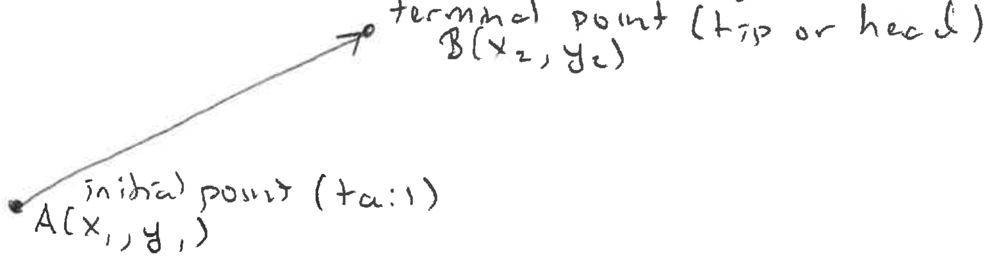

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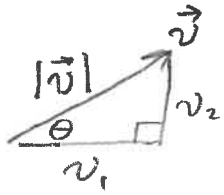
Consider $\vec{v} = \langle v_1, v_2 \rangle$. We define the angle θ as that between the vector and the x-axis



$$\tan\theta = \frac{v_2}{v_1} \Rightarrow \theta = \tan^{-1}\frac{v_2}{v_1}$$

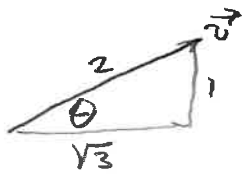
$$\cos\theta = \frac{v_1}{|\vec{v}|} \qquad \sin\theta = \frac{v_2}{|\vec{v}|}$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \overset{1}{\sin\frac{\pi}{2}}\cos\theta - \overset{0}{\cos\frac{\pi}{2}}\sin\theta = \cos$$

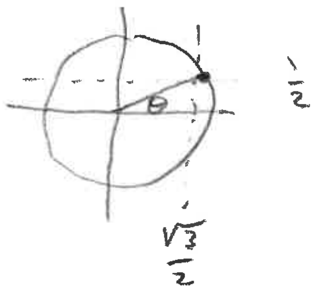
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Example: Let $\vec{v} = \langle \sqrt{3}, 1 \rangle$. Find the length and the angle θ.

Solution: $|\vec{v}| = \sqrt{\sqrt{3}^2 + 1^2} = 2$



$$\tan\theta = \frac{1}{\sqrt{3}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$



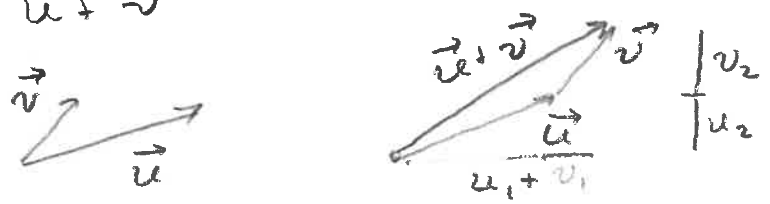
$$\theta = \frac{\pi}{6} = 30^\circ$$

Operations

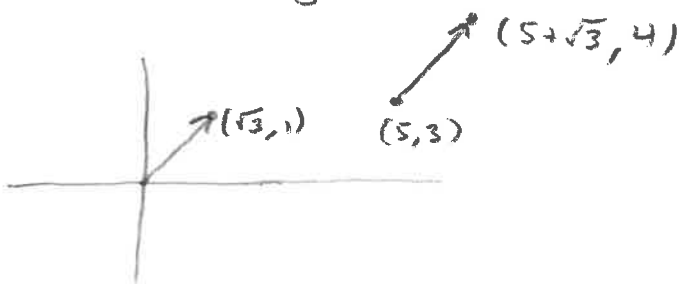
1. Vector Addition $\vec{u} + \vec{v}$
2. Scalar multiplication $c\vec{v}$ $c \in \mathbb{R}$ (unit vector)
3. dot product $\vec{u} \cdot \vec{v}$
4. cross product $\vec{u} \times \vec{v}$
5. subtraction

Two vectors sum according to the triangle rule or the parallelogram rule.

Triangle rule
$\vec{u} + \vec{v}$

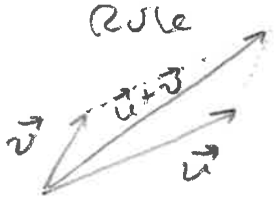


Note that although vectors have magnitude and direction they do not have position.



The components of $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$
Vectors sum componentwise.

Parallelogram Rule



Scalar multiplication multiplies a real number c by each component of the vector, thereby scaling the vector.

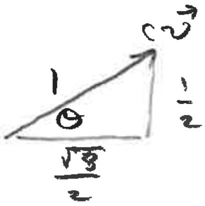
$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$$

$$|c\vec{v}| = \sqrt{(cv_1)^2 + (cv_2)^2 + (cv_3)^2} = \sqrt{c^2(v_1^2 + v_2^2 + v_3^2)}$$

$$= |c|\sqrt{v_1^2 + v_2^2 + v_3^2} = |c|\,|\vec{v}|$$

Example: Let $\vec{v} = \langle \sqrt{3}, 1 \rangle$ and $c = \frac{1}{2}$.
Find the length and angle of $c\vec{v}$.

$$c\vec{v} = \frac{1}{2}\langle \sqrt{3}, 1 \rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$



$$|c\vec{v}| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}}$$

$$= 1$$

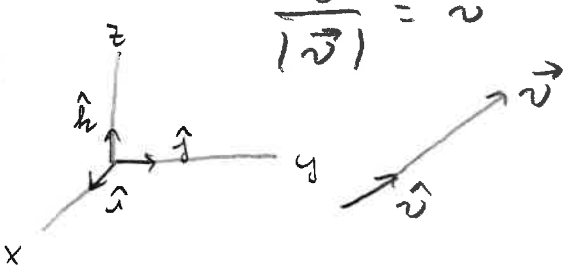
$$\sin\theta = \frac{\frac{1}{2}}{1} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

A unit vector $\hat{v}$ has length $|\hat{v}| = 1$.

The zero vector is $\vec{0} = \langle 0, 0, 0 \rangle$

Any nonzero vector can be made a unit vector by dividing by its length.

$$\frac{\vec{v}}{|\vec{v}|} = \hat{v} \Rightarrow \vec{v} = \underbrace{|\vec{v}|}_{\text{length}}\,\underbrace{\hat{v}}_{\text{direction}}$$



There are three special unit vectors:

$\hat{i} = \langle 1, 0, 0 \rangle$
$\hat{j} = \langle 0, 1, 0 \rangle$
$\hat{k} = \langle 0, 0, 1 \rangle$