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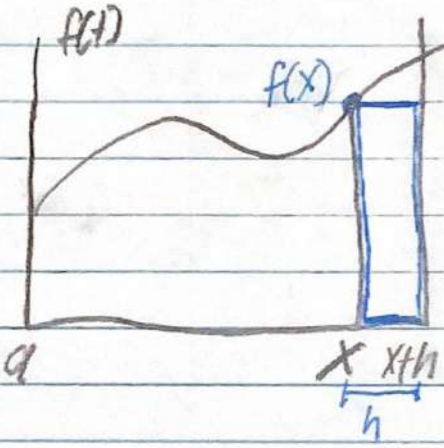
Math 142
08/31/22

5.3 – The Fundamental Theorem of calculus

How to find limits of Riemann Sums?
– Summation, limit tricky

Def. The area function for $f(x)$ with left end point a

$$A(x) = \int_a^x f(t)\,dt$$



Claim: $A(x)$ is an antid. for $f(x)$. Why?

$A(x+h) - A(x) \approx h \cdot f(x)$

$$f(x) \approx \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

FTC (part 1)

If f(x) is continous on [a,b], then A(x) is an antid. of f(x):

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

ex Find $\frac{dy}{dx}$ for $y = \int_4^x \underbrace{(t^3+5t+2)}_{f(t)}\,dt$

$$\frac{dy}{dx} = x^3+5x+2$$

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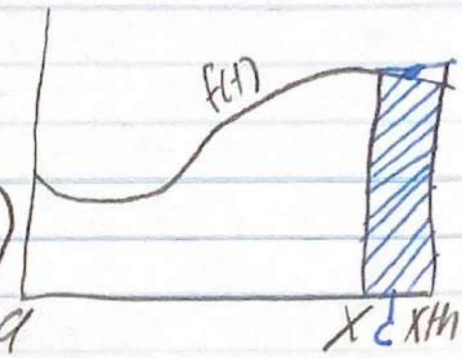
FTC part 1 (con.)

If $f(x)$ is cont. on $[a,b]$, Then

$A(x) = \int_a^x f(t)\,dt$ is an antid. of $f(x)$.

Proof. $A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$



$= \lim_{h \to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right)$

$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$

By MVT for integrals, There c in $[x, x+h]$ with $\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c)$. As $h \to 0$, $c \to x$,

So $f(c) \to f(x)$. Thus $A'(x) = \lim_{h \to 0} f(c) = f(x)$.

ex. Find $\frac{dy}{dx}$ for $y = \int_x^2 \sin(\sin t)\,dt$

$= -\int_2^x \sin(\sin t)\,dt$

$\frac{dy}{dx} = \boxed{-\sin(\sin x)}$

ex $y(x) = \int_0^{x^3} t e^{t^2}\,dt$

$y(u) = \int_0^u t e^{t^2}\,dt, \; u = x^3$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= u e^{u^2} \cdot 3x^2$

$= x^3 e^{x^6} \cdot 3x^2$

$\boxed{= 3x^5 e^{x^6}}$

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FTC part 2

If $f(x)$ is integrable on $[a,b]$, and $F(x)$ is any antid. of $f(x)$ on $[a,b]$,

$$\int_a^b f(x)\,dx = F(b)-F(a) = F(x)\Big|_a^b$$

Proof. By part 1, f has antid. $A(x)=\int_a^x f(t)\,dt$. Any antid. of f satisfies

$F(x)=A(x)+C$. Then $F(b)-F(a)$

$= (A(b)\cancel{+C}) - (A(a)\cancel{+C})$

$= A(b)-A(a)$

$= \int_a^b f(t)\,dt - \cancel{\int_a^a f(t)\,dt}^{\,0}$

$= \int_a^b f(x)\,dx$

ex $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

$\int_{-3}^{-2} \frac{3}{x^2}\,dx = -\frac{3}{x}\Big|_{-3}^{-2} = \frac{3}{2} - 1 = \frac{1}{2}$

$\int_0^{\frac{\pi}{3}} 2\sec^2 x\,dx = 2\tan x\Big|_0^{\frac{\pi}{3}} = 2(\sqrt{3}) - 2(0) = 2\sqrt{3}$

$\int_0^{\ln 2} e^{3x}\,dx = \frac{1}{3}e^{3x}\Big|_0^{\ln 2} = \frac{1}{3}(8-1) = \frac{7}{3}$

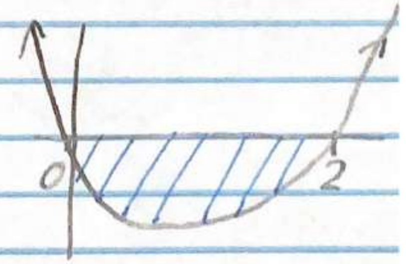
$e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$

$\int_0^1 \frac{2}{1+x^2}\,dx = 2\arctan x\Big|_0^1 = 2\frac{\pi}{4} - 0 = \frac{\pi}{2}$

$\int_0^2 x(x-2)\,dx = \int_0^2 (x^2-2x)\,dx = \frac{x^3}{3} - x^2\Big|_0^2$

$= \left(\frac{2^3}{3} - 2^2\right) - \left(\frac{0^3}{3} - 0^2\right) = \frac{8}{3} - 4 = -\frac{4}{3}$

↑
Area below x-Axis.



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FTC part 2 (con.)

$$\int_1^{\sqrt{2}} \frac{x^2+\sqrt{x}}{x^2}\,dx$$

$$= \int_1^{\sqrt{2}} (1 + x^{-3/2})\,dx = x - 2x^{-\frac{1}{2}} \Big|_1^{\sqrt{2}}$$

$$(\sqrt{2} - 2(2^{1/2})^{-1/2}) - (1 - \frac{2}{1^2}) = \sqrt{2} - 2^{3/4} + 1$$

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FTC part 2 (con.)

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

$$\int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\,dx$$

$$=\int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx$$

$$=\int_0^{\frac{\pi}{4}} (\sec^2 x+\sec x\cdot\tan x)\,dx$$

$$=(\tan x+\sec x)\Big|_0^{\pi/4}$$

$$=(1+\sqrt{2})-(0+1)=\sqrt{2}$$

$$\int_1^3 \frac{(2+x)(2-x)}{x}\,dx = \int_1^3 \frac{4-x^2}{x}\,dx$$

$$= \int_1^3 \left(\frac{4}{x} - x\right)dx = \left(4\ln|x| - \frac{x^2}{2}\right)\Big|_1^3$$

$$= \left(4\ln|3| - \frac{9}{2}\right) - \left(4\cdot 0 - \frac{1}{2}\right) = 4\ln|3| - 4$$

Derivative and Integral are (Almost) inverses!

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^x \frac{d}{dt} F(t)\,dt = F(x) - F(a)$$

Net change Theorem

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$