Probability and Random Variables (ECE313)

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Weak law of large numbers+central limit theorem
Fall 2023

Weak law of large numbers-Central limit Theorem

▷ Weak law of large numbers:

• Probability computations are easy to deal if we have one or two random variables.

• Probability problems becomes computationally intractable if we are dealing, let's say, with 100 random variables.

• All formulas of probability that we have still apply, but they involve summations over large range of combinations.

$\rightarrow$ We can introduce limits that can simplify a lot of computations

Weak law of large numbers-Central limit Theorem

▷ Markov inequality: $\boxed{X \geq 0; \quad \mathbb{E}[X] \geq a\mathbb{P}(X \geq a)}$ or $\boxed{X \geq 0; \quad \mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}}$

- **Proof:**

$$X \geq 0; \quad \mathbb{E}[X] = \sum_{x \geq 0} x\mathbb{P}_X(x) \geq \sum_{x \geq a} x\mathbb{P}_X(x) \geq \sum_{x \geq a} a\mathbb{P}_X(x) = a\sum_{x \geq a} \mathbb{P}_X(x) = a\mathbb{P}_X(x \geq a)$$

▷ Chebytshev's inequality: $\boxed{\mathbb{P}_X(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}}$, $\quad \mu = \mathbb{E}[X]$ and $\sigma^2 = V(X)$

- **Proof:**

$$V(X) = \mathbb{E}[(X-\mu)^2] \geq a^2\mathbb{P}_X((X-\mu)^2 \geq a) = a^2\mathbb{P}_X(|x-\mu| \geq a) \Rightarrow \mathbb{P}_X(|X-\mu| \geq a) \leq \frac{V(X)}{a^2}$$

▷ Convergence in probability: Let a sequence of random variables Y_n that converge to a. So, we have

$$Y_n \underset{n \to \infty}{\rightarrow} a \Leftrightarrow \lim_{n \to \infty} \mathbb{P}(|Y_n - a| > \varepsilon) = 0$$

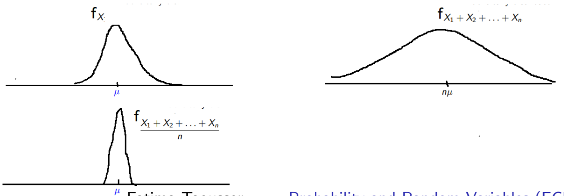
Weak law of large numbers-Central limit Theorem

▷ Convergence of the sample mean (weak law of large numbers):

Let $X_1, X_2, \ldots, X_n$ be n random variables which are independent and identically distributed (i.i.d) with a mean μ and a standard deviation σ^2. Let the sum $S_n = X_1 + X_2 + \ldots + X_n$. Consider now the mean of this sample of random variables

$$M_n = \frac{X_1 + X_2 + \ldots + X_n}{n} = \frac{S_n}{n}$$

- $\mathbb{E}[M_n] = \frac{\mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]}{n} = \frac{n\mu}{n} = \mu$
- $V(M_n) = \frac{V(X_1) + V(X_2) + \ldots + V(X_n)}{n} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$ $\rightarrow$ The variance becomes smaller for a large sample size.



Weak law of large numbers-Central limit Theorem

▷ Applying the Chebyshev's inequality:

$$\mathbb{P}(|M_n - \mu| \geq \varepsilon) \leq \frac{V(M_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \underset{n\to\infty}{\to} \varepsilon' \Rightarrow M_n \underset{n\to\infty}{\to} \mu \text{ in probability}$$

⇒ The sample mean converge to the true mean.

▷ Example:

Let $X_i = \begin{cases} 1 \\ 0 \end{cases} \to \mu_i = \frac{1}{2}, \;\; \sigma^2 = \mathbb{E}[X_i^2] - \mu^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Let $M_n = \frac{X_1 + X_2 + \ldots + X_n}{n} = f$. Goal: 95% confidence of 1% error

$$\mathbb{P}(|M_n - f| \geq 0.01) \leq 0.05 \text{ ?}$$

- Applying the Chebyshev's inequality:

$$\mathbb{P}(|M_n - f| \geq 0.01) \leq \frac{\sigma^2}{n(0.01)^2} \leq \frac{1}{4n(0.01)^2} \leq 0.05 \Rightarrow n \geq \frac{1}{4(0.05)(0.01)^2} = 50000.$$

⇒ $n = 50000$ is a large size → We can solve this problem using the central limit theorem.

Weak law of large numbers-Central limit Theorem

▷ Let us scale the sum $S_n = X_1 + X_2 + \ldots + X_n$ by $\sqrt{n}$.

$$\Rightarrow M_n = \frac{\sqrt{n} S_n}{n} = \frac{S_n}{\sqrt{n}} = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}$$

- $\mathbb{E}[M_n] = \dfrac{\mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]}{\sqrt{n}} = \dfrac{n\mu}{\sqrt{n}} = \sqrt{n}\mu$
- $V(M_n) = V(\dfrac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}) = \dfrac{V(X_1) + V(X_2) + \ldots + V(X_n)}{n} = \dfrac{n\sigma^2}{n} = \sigma^2 \rightarrow$ If n change, the variance doesn't change. Let the random variable

$$Z_n = \frac{S_n - \mathbb{E}[S_n]}{\sqrt{V(S_n)}} = \frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}\sigma} \Rightarrow \begin{cases} \mathbb{E}[Z_n] = \frac{\mathbb{E}[S_n] - \mathbb{E}[S_n]}{\sqrt{n}\sigma} = 0 \\ V(Z_n) = V(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}\sigma}) = \frac{V(S_n)}{n\sigma^2} = \frac{n\sigma^2}{n\sigma^2} = 1 \end{cases}$$

▷ Central limit theorem 1: Let Z_n be any distribution defined as above and Let $Z \rightarrow \mathcal{N}(0,1)$. Then

$$\mathbb{P}(Z_n \leq c) \underset{n \to \infty}{\rightarrow} \mathbb{P}(Z \leq c), \quad \text{for every c} \quad (\text{i.e; } Z_n = \frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}\sigma} \rightarrow \mathcal{N}(0,1))$$

$\rightarrow$ Form $\mathbb{P}(Z_n \leq c)$ we can deduce the probability $\mathbb{P}(S_n \leq c)$

Central limit Theorem

▷ Apply to Binomial distributions:

- Let X_i be a Bernoulli distributions with parameter (p), $0 < p < 1$
- Let $S_n = X_1 + X_2 + X_3 + ... + X_n =$ Binomial (n,p)
- The mean of $S_n = \mathbb{E}[S_n] = np$
- The variance of $S_n = V(S_n) = np(1-p)$.
- The distribution $Z_n = \dfrac{S_n - np}{\sqrt{np(1-p)}}$ follows a standard normal distribution
- Let $n = 36,\ p = 0.5 \Rightarrow \mathbb{E}[S_n] = np = 18$, $V(S_n) = \sigma^2 = np(1-p) = 9$.
- Find $\mathbb{P}(S_n \leq 21)$?

$$\mathbb{P}(S_n \leq 21) = \mathbb{P}\left(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}\sigma} \leq \frac{21-18}{\sqrt{9}}\right) = \mathbb{P}(Z_n \leq 1) = 0.843 \rightarrow Z_n \rightarrow \mathcal{N}(0,1)$$

▷ Exact answer:

$S_n \rightarrow \mathcal{B}(n,p) \Rightarrow P(S_n \leq 21) = \sum_{k=0}^{21} C_k^{36}(0.5)^k(0.5)^{36-k} = \sum_{k=0}^{21} C_k^{36}(0.5)^{36} = 0.8785$

- Remark:

In general $n \approx 15$ gives a good approximation.

Central limit Theorem

Remark 2: Since S_n is a discrete random variable, so we can do some compromise to compute $\mathbb{P}(S_n = k)$.
- Let the previous example s.t $n = 36$, $p = 0.5 \Rightarrow$
$\mathbb{E}[S_n] = np = 18$, $V(S_n) = \sigma^2 = np(1-p) = 9$.

$$\mathbb{P}(S_n = 19) = \mathbb{P}(18.5 \leq S_n \leq 19.5) = \mathbb{P}(\frac{18.5 - 18}{\sqrt{9}} \leq \frac{S_n - 18}{\sqrt{9}} \leq \frac{19.5 - 18}{\sqrt{9}})$$

$$= \mathbb{P}(0.17 \leq Z \leq 0.5) = \mathbb{P}(Z \leq 0.5) - \mathbb{P}(Z \leq 0.17) = 0.124 \approx \mathbb{P}(S_n = 19)$$

▷ Exact answer:

$$S_n \to \mathcal{B}(30, 0.5) \Rightarrow P(S_n = 19) = C_{19}^{36}(0.5)^{19}(0.5)^{36-19} = 0.1251$$

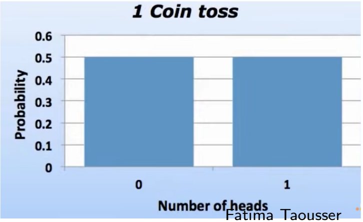
Central limit Theorem

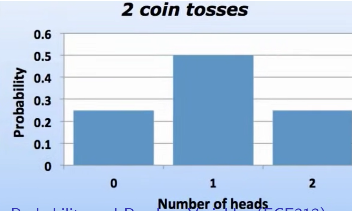
- Example 2:
- Experiment1: Filliping a fair coin one time $\rightarrow$ {H,T}.
- The random variable X: {The number of Heads}

$$\mathbb{P}(X=0)=\frac{1}{2}, \quad \mathbb{P}(X=1)=\frac{1}{2} \rightarrow \text{Bernoulli } (0.5)$$

- Experiment2: Filliping a fair coin two times $\rightarrow$ {HH,HT, TH, TT}.
- The random variable X: {The number of Heads} $\rightarrow X_1 + X_2 \rightarrow \mathcal{B}(2, 0.5)$

$$\mathbb{P}(X=0)=\frac{1}{4}, \quad \mathbb{P}(X=1)=\frac{2}{4}, \quad \mathbb{P}(X=2)=\frac{1}{4}$$

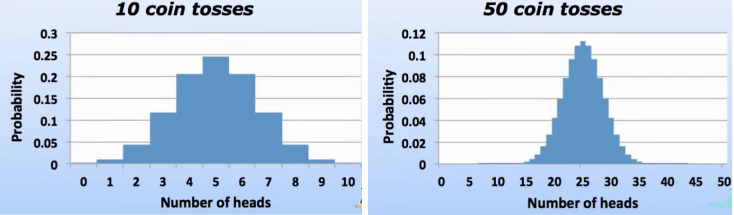




Central limit Theorem

- Experiment3: Filliping a fair coin 10 times $\rightarrow \{HTHHHTTHTT, \ldots\}$
- The random variable X: {The number of Heads} $\rightarrow X_1 + \ldots + X_{10} \rightarrow \mathcal{B}(10, 0.5)$

- Experiment4: Filliping a fair coin 50 times.
- The random variable X: {The number of Heads} $\rightarrow X_1 + \ldots + X_{50} \rightarrow \mathcal{B}(50, 0.5)$



- Let $S_n = X_1 + \ldots + X_{50} \rightarrow \mathcal{B}(50, 0.5)$, $\mathbb{E}[S_n] = np = (50).(0.5) = 25$,
$V(S_n) = \sigma^2 = np(1-p) = (25).(0.5) = 12.5$

$$\mathbb{P}(S_n \leq 19) = \mathbb{P}(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}\sigma} \leq \frac{19-25}{\sqrt{12.5}}) = \mathbb{P}(Z_n \leq -1.69) = 0.046 \rightarrow Z_n \rightarrow \mathcal{N}(0,1)$$

Central limit Theorem

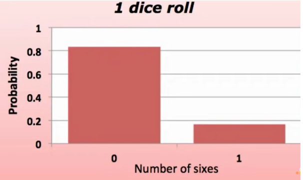
- Example 3:
- Experiment1: Rolling a 6 sided die one time $\rightarrow \{1,2,3,4,5,6\}$.
- The random variable X: {The number of sixes}

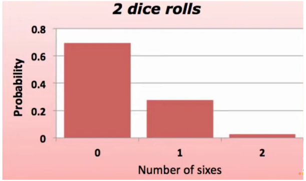
$$\mathbb{P}(X=0)=\frac{5}{6}, \quad \mathbb{P}(X=1)=\frac{1}{6} \rightarrow \text{Bernoulli } \left(\frac{1}{6}\right)$$

- Experiment2: Rolling a 6 sided die two times $\rightarrow \{(1,2),(1,4),\dots\} = 36$ elements.

- The random variable X: {The number of sixes} $\rightarrow X_1 + X_2 \rightarrow \mathcal{B}(2, \frac{1}{6})$

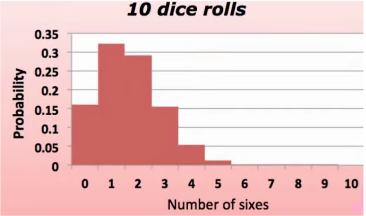
$$\mathbb{P}(X=0)=\frac{25}{36}=0.69, \quad \mathbb{P}(X=1)=\frac{10}{36}=0.27, \quad \mathbb{P}(X=2)=\frac{1}{36}=0.027$$

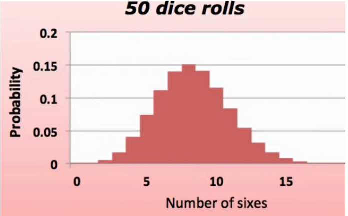




Continuous Random Variables-Central limit Theorem

- Experiment3: Rolling a 6 sided die 10 times .
- The random variable X: {The number of sixes} $\rightarrow X_1 + \ldots + X_{10} \rightarrow \mathcal{B}(10, \frac{1}{6})$
- Experiment4: Rolling a 6 sided die 50 times.
- The random variable X: {The number of sixes} $\rightarrow X_1 + \ldots + X_{50} \rightarrow \mathcal{B}(50, \frac{1}{6})$





- Let $S_n = X_1 + \ldots + X_{50} \rightarrow \mathcal{B}(50, \frac{1}{6})$, $\mathbb{E}[S_n] = np = (50).(\frac{1}{6}) = 8.33$, $V(S_n) = \sigma^2 = np(1-p) = (8.33).(1 - \frac{1}{6}) = 6.94$

$$\mathbb{P}(S_n \leq 15) = \mathbb{P}(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}\sigma} \leq \frac{15 - 8.33}{\sqrt{6.94}}) = \mathbb{P}(Z_n \leq 2.53) = 0.99 \rightarrow Z_n \rightarrow \mathcal{N}(0,1)$$

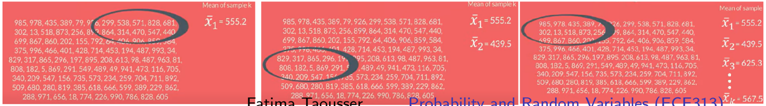
Weak law of large numbers-Central limit Theorem

▷ Second version: Let a data set Y that follow a random distribution with mean μ and standard deviation σ $\rightarrow Y(\mu, \sigma^2)$

- Let us take k subsets of this data set with size n (sampling) and compute the mean and the variance of each sample subset

$$\underbrace{\{x_1, x_2, \ldots, x_n\}}_{X_1 \rightarrow (\mu_1, \sigma_1)}, \underbrace{\{x_1^{'}, x_2^{'}, \ldots, x_n^{'}\}}_{X_2 \rightarrow (\mu_2, \sigma_2)}, \ \ldots \ , \underbrace{\{x_1^{*}, x_2^{*}, \ldots, x_n^{*}\}}_{X_k \rightarrow (\mu_k, \sigma_k)},$$

$$X = \{X_1, X_2, \ldots, X_k\} \underbrace{\rightarrow}_{\text{means}} \bar{X} = \{\mu_1, \mu_2, \ldots \mu_k\} \rightarrow \bar{X}(\bar{\mu}, \sigma_{\bar{X}})$$



Central limit Theorem

- The central limit Theorem 2:

- The mean of sample means equal to the mean of the overall data set Y

$$\Rightarrow \mathbb{E}(\bar{X}) = \mathbb{E}(Y) \;\Rightarrow\; \bar{\mu} = \mu \;\text{ (it is independent to the sample size).}$$

- The standard deviation of sample means

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

- The central limit Theorem 3:

If the data set is **normally distributed** then the sample means will have a normal distribution (it is independent to the sample size) $\rightarrow \; \bar{X} \rightarrow \mathcal{N}(\bar{\mu}, \sigma^2_{\bar{X}})$

- The central limit Theorem 4:

If the data set is **not normally distributed** but the sample size is "$n \geq 30$", then the sample means will approximate a normal distribution $\rightarrow \; \bar{X} \rightarrow \mathcal{N}(\bar{\mu}, \sigma^2_{\bar{X}})$ for $n \geq 30$

Central limit Theorem

- Example1:

Suppose salaries at a very large corporation have a mean of \$62000 and a standard deviation of \$32000. If 100 employees are randomly selected, what is the probability that their average salary exceeds \$66000?

Solution:

- Let Y be the data set of all the salaries such that $\mu = \$62000$ and $\sigma = \$32000$
$\Rightarrow \ Y \rightarrow (\$62000, (\$32000)^2)$ (We don't have any information on how these salaries are distributed!!).

- Let $\bar{X}$ be the set of averages salaries of groups of 100 employees $\Rightarrow \ \bar{X} \rightarrow \ \mathcal{N}(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2)$

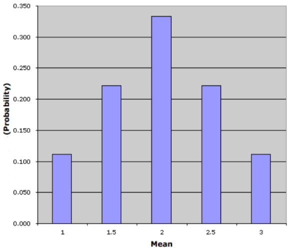
$$\mathbb{P}(\bar{X} > 66000) = \mathbb{P}(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > \frac{66000 - 62000}{\frac{32000}{\sqrt{100}}}) = \mathbb{P}(Z > 1.25) = 1 - \mathbb{P}(Z < 1.25)$$

with $Z \rightarrow \mathcal{N}(0, 1) \Rightarrow \mathbb{P}(\bar{X} > 66000) = 1 - \mathbb{P}(Z < 1.25) = 1 - 0.8944 = 0.1056 \rightarrow$ (using the table of the standard normal distribution)

Central limit Theorem

- Example 2: Consider three pool balls, each with a number on it $\rightarrow X = \{1, 2, 3\}$. Two of the balls are selected randomly (with replacement), and the average of their numbers is computed. All possible outcomes are:
{(1,1), (1,2), (1,3), (2,3), (3,3), (2,2), (2,1), (2,3), (3,1) }. means of all these outcomes are $\bar{X} = \{1, \frac{3}{2}, 2, \frac{5}{2}, 3, 2, \frac{3}{2}, \frac{5}{2}, 2\}$, and shown below:

Mean	Frequency	probability
1.0	1	0.111
1.5	2	0.222
2.0	3	0.333
2.5	2	0.222
3.0	1	0.111



$$\mathbb{E}[\bar{X}] = \frac{1 + 1.5 + 2 + 2.5 + 3 + 2 + 1.5 + 2.5 + 2}{9} = 2 = \mathbb{E}[X] = \frac{1+2+3}{3}$$

$$V(\bar{X}) = \mathbb{E}[\bar{X}^2] - \mathbb{E}^2[\bar{X}] = \frac{1}{3} < V(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X] = \frac{2}{3}$$

Central limit Theorem

- Example2:

A certain group of welfare recipients receives SNAP benefits of \$110 per week, in average, with a standard deviation of \$20. Knowing that these benefits follow a normal distribution, what is the probability that the mean of a random sample of 25 people will be less than \$120 per week?

Solution:

- Let Y be the data set of all benefits such that $\mu = \$110$ and $\sigma = \$20$
$\Rightarrow \; Y \rightarrow \mathcal{N}(\$110, (\$20)^2)$.

- Let $\bar{X}$ be the set of averages benefits of groups of 25 persons $\Rightarrow \; \bar{X} \rightarrow \; \mathcal{N}(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2)$

$$\mathbb{P}(\bar{X} < 120) = \mathbb{P}(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{120 - 110}{\frac{20}{\sqrt{25}}}) = \mathbb{P}(Z < 2.5)$$

with $Z \rightarrow \mathcal{N}(0, 1) \Rightarrow \mathbb{P}(\bar{X} < 120) = \mathbb{P}(Z < 2.5) = 0.9938 \rightarrow$ (using the table of the standard normal distribution)

Central limit Theorem

- Example3:

Suppose the grades in a finite mathematics class are Normally distributed with a mean of 75 and a standard deviation of 5.

a) What is the probability that a randomly selected student had a grade of at least 83?

b) What is the probability that the average grade for 5 randomly selected students was at least 83?

Solution:

- Let Y be the data set of all scores such that $\mu = 75$ and $\sigma = 5 \;\Rightarrow\; Y \rightarrow \mathcal{N}(75, 5^2)$.

a) $\mathbb{P}(X > 83) = \mathbb{P}(\frac{X-\mu}{\sigma} > \frac{83-75}{5}) = \mathbb{P}(Z > 1.6) = 1 - \mathbb{P}(Z < 1.6) = 1 - 0.952 = 0.054$

b) Let $\bar{X}$ be the set of averages scores of groups of 5 persons $\;\Rightarrow\; \bar{X} \rightarrow \; \mathcal{N}(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2)$

$$\mathbb{P}(\bar{X} > 83) = \mathbb{P}(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} > \frac{83 - 75}{\frac{5}{\sqrt{5}}}) = \mathbb{P}(Z > 3.57) = 1 - \mathbb{P}(Z < 3.57) = 1 - 0.9998 = 0.0002$$

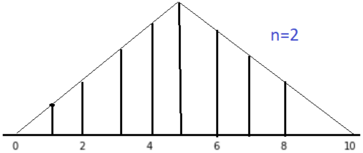
Central limit Theorem

- Example 4:

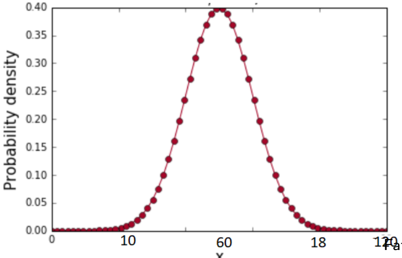
Let $X = \mathcal{U}(1,5)$.

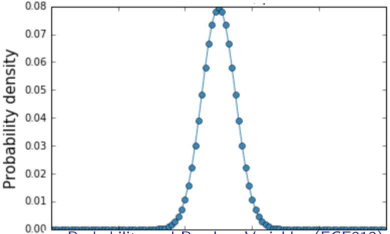
n=1

n=2



N=10





Central limit Theorem

- The importance of the central limit Theorem:
  - Universal and easy to apply, only means and variances matter.
  - Simplify the study of a large set of data $\rightarrow$ Fairly accurate computational shortcut.
  - If the data are randomly distributed, so by taking samples greater than 30 allows us to get many information about the data set and solve many problems.

Remark: The central limit theorem cannot be applied for distributions which are not independent and identically distributed (i.i.d)

Central limit Theorem-Training

Problem 1: The amount of impurity in a batch of a chemical product is a random variable with mean value $\mu = 9$ g and standard deviation $\sigma = 1.5$ g. (unknown distribution) If 50 batches are independently prepared, what is the (approximate) probability that the average amount of impurity in these 50 batches is between 8.6 and 9.5 g?

Solution: Let $\bar{X} = \{\mu_1, \mu_2, \ldots, \mu_k\}$ where μ_i are the means of samples with size $n = 50$. According to the central limit theorem $\bar{X} \to \mathcal{N}(\mu, \frac{\sigma}{\sqrt{n}})$

$$\mathbb{P}(8.6 \leq \bar{X} \leq 9.5) = \mathbb{P}\left(\frac{8.6-9}{\frac{1.5}{\sqrt{50}}} \leq \frac{\bar{X}-9}{\frac{1.5}{\sqrt{50}}} \leq \frac{9.5-9}{\frac{1.5}{\sqrt{50}}}\right) = \mathbb{P}(-1.88 \leq Z \leq 2.35)$$

$$= \mathbb{P}(Z \leq 2.35) - \mathbb{P}(Z \leq -1.88) = 0.9906 - 0.03 = 0.96.$$

- If we will decrease the size of the sample $n = 10$

$$\mathbb{P}(8.6 \leq \bar{X} \leq 9.5) = \mathbb{P}\left(\frac{8.6-9}{\frac{1.5}{\sqrt{10}}} \leq \frac{\bar{X}-9}{\frac{1.5}{\sqrt{10}}} \leq \frac{9.5-9}{\frac{1.5}{\sqrt{10}}}\right) = \mathbb{P}(-0.84 \leq Z \leq 1.05) = 0.85 - 0.2 = 0.65$$

- By increasing n the probability that the means of samples be close to the mean of the overall set becomes less since the variance becomes smaller.

Central limit Theorem-Training

Problem 2:

An unknown distribution X_i has a mean of $\mu = 90$ and a standard deviation of $\sigma = 15$. A sample of size $n = 80$ is drawn randomly from the population.

a. Find the probability that the sum of the 80 values (or the total of the 80 values) is more than 7500.

b. Find the sum that is 1.5 standard deviations below the mean of the sums

Solution:

We want to compute the probability for the sum of 80 values $S_n = \sum_{i=1}^{80} X_i \to \mathcal{N}(n\mu, (\sqrt{n}\sigma)^2) = \mathcal{N}(80(90), (15\sqrt{80})^2)$ (according to the central limit theorem)

$$\mathbb{P}(S_n > 7500) = \mathbb{P}(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{V(S_n)}} > \frac{7500 - 80(90)}{15\sqrt{80}}) = \mathbb{P}(Z > 2.23) = 1 - \mathbb{P}(Z < 2.23)$$

$$= 1 - 0.987 = 0.013$$

Central limit Theorem-Training

Problem 3:

Suppose a surveyor wants to measure a known distance, say of 1 mile, using a transit and some method of triangulation. He knows that because of possible motion of the transit, atmospheric distortions, and human error, any one measurement is apt to be slightly in error. He plans to make several measurements and take an average. He assumes that his measurements are independent random variables with a common distribution of mean $\mu = 1$ and standard deviation $\sigma = 0.0002$.

- What can he say about the average?
- How many measurements should he make to be reasonably sure that his average lies within 0.0001 of the true value (let say 95% confidence)?

Solution:

- He can say that if n is large, the average $\frac{S_n}{n}$ has a density function that is approximately normal, with mean $\mu = 1$ mile, and standard deviation $\sigma = \frac{0.0002}{\sqrt{n}}$ miles.
- According to the Chebyshev's inequality, we have

$$\mathbb{P}(|\frac{S_n}{n} - \mu| \geq 0.0001) \leq \frac{V(\frac{S_n}{n})}{(0.0001)^2} = \frac{(0.0002)^2}{n.10^{-8}} = \frac{4}{n} \leq 0.05 \Rightarrow n \geq 80$$