# Probability and Random Variables (ECE313/ECE317)

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# Multivariate Random Variables

- Multivariate random variables: accommodate the dependence between two or more random variables.
- The concepts in the multivariate random variables (such as expectations and variance) are analogous to those under uni-variate random variables.
- Bivariate random variable: accommodate the dependence between two random variables $X_1$ and $X_2$.
- The PMF or PDF for a bivariate random variable gives the probability that the two random variables each take a certain value.
- The joint Probability Mass Function (PMF) is given by

$$\mathbb{P}_{X_1,X_2}(x_1, x_2) = \mathbb{P}(X_1 = x_1 \underbrace{\boxed{;}}_{\text{and}} X_2 = x_2)$$

such that

1) $0 \leq \mathbb{P}_{X_1,X_2} \leq 1$
2) $\sum_{x_1} \sum_{x_2} \mathbb{P}_{X_1,X_2}(x_1, x_2) = 1$

# Multivariate Random Variables

- Example 1: Consider the experiment of tossing a fair coin three times, and independently of the first coin, we toss a second fair coin three times. Let
$X_1 =$ The number of Heads for the first coin
$X_2 =$ The number of Tails for the second coin
- The two coins are tossed independently, so for any pair of possible values $(x_1, x_2)$ of $X_1$ and $X_2$ we have the following joint probability,

$$\begin{aligned}
\mathbb{P}_{X_1,X_2}(x_1, x_2) &= \mathbb{P}(X_1 = x_1 \text{ and } X_2 = x_2) \\
&= \mathbb{P}(\{X_1 = x_1\} \cap \{X_2 = x_2\}) \\
&= \mathbb{P}(X_1 = x_1).\mathbb{P}(X_2 = x_2) \\
&= \mathbb{P}_{X_1}(x_1).\mathbb{P}_{X_2}(x_2)
\end{aligned}$$

# Multivariate Random Variables

$\Omega_1 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$\Omega_2 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$X =$ The number of Heads for the first coin $= \{0, 1, 2, 3\}$
$Y =$ The number of Tails for the second coin $= \{0, 1, 2, 3\}$

| $x_i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mathbb{P}_X(x_i)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

| $x_i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mathbb{P}_Y(y_i)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

$$\mathbb{P}(X = 3; Y = 2) = \mathbb{P}_X(3).\mathbb{P}_Y(2) = \frac{1}{8}.\frac{3}{8} = \frac{3}{64}$$

- We can compute all the probabilities in the following table

| $x_i/y_i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\frac{1}{64}$ | $\frac{3}{64}$ | $\frac{3}{64}$ | $\frac{1}{64}$ |
| 1 | $\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}$ |
| 2 | $\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}$ |
| 3 | $\frac{1}{64}$ | $\frac{3}{64}$ | $\frac{3}{64}$ | $\frac{1}{64}$ |

# Multivariate Random Variables

$$\sum_X \sum_Y \mathbb{P}_{X,Y} = \sum_X (\sum_Y \mathbb{P}_{X,Y}) = \sum_X (\sum_Y \mathbb{P}_X . \mathbb{P}_Y) = \sum_X \mathbb{P}_X (\sum_Y \mathbb{P}_Y) = 1$$

| $x_i/y_i$ | 0 | 1 | 2 | 3 | $\sum_{y_i} \mathbb{P}(X = x_i; Y = y_i)$ |
|---|---|---|---|---|---|
| 0 | $\frac{1}{64}$ | $\frac{3}{64}$ | $\frac{3}{64}$ | $\frac{1}{64}$ | $\frac{8}{64}$ |
| 1 | $\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}$ | $\frac{24}{64}$ |
| 2 | $\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}$ | $\frac{24}{64}$ |
| 3 | $\frac{1}{64}$ | $\frac{3}{64}$ | $\frac{3}{64}$ | $\frac{1}{64}$ | $\frac{8}{64}$ |
| $\sum_{x_i} \mathbb{P}(X = x_i; Y = y_i)$ | $\frac{8}{64}$ | $\frac{24}{64}$ | $\frac{24}{64}$ | $\frac{8}{64}$ | $\sum_i \mathbb{P}_{X,Y} = 1$ |

- $\sum_{x_i,y_i} \mathbb{P}_{X,Y} = \sum_{x_i} [\sum_{y_i} \mathbb{P}(X = x_i; Y = y_i)] = \sum_{x_i} \mathbb{P}(X = x_i, Y = 0) + \sum_{x_i} \mathbb{P}(X = x_i, Y = 1) + \sum_{x_i} \mathbb{P}(X = x_i, Y = 2) + \sum_{x_i} \mathbb{P}(X = x_i, Y = 3)] = \dfrac{8}{64} + \dfrac{24}{64} + \dfrac{24}{64} + \dfrac{8}{64} = \dfrac{64}{64} = 1$
- $\sum_{x_i,y_i} \mathbb{P}_{X,Y} = \sum_{y_i} [\sum_{x_i} \mathbb{P}(X = x_i; Y = y_i)] = \sum_{y_i} \mathbb{P}(X = 0, Y = y_i) + \sum_{y_i} \mathbb{P}(X = 1, Y = y_i) + \sum_{y_i} \mathbb{P}(X = 2, Y = y_i) + \sum_{y_i} \mathbb{P}(X = 3, Y = y_i)] = \dfrac{8}{64} + \dfrac{24}{64} + \dfrac{24}{64} + \dfrac{8}{64} = \dfrac{64}{64} = 1$

# Multivariate Random Variables

- If $\Omega_1$ and $\Omega_2$ are the sample sets of the random variables $X_1$ and $X_2$ respectively $\Rightarrow$ The sample set of the bivariate random variable $(X_1, X_2)$ is given by $\Omega_1 \times \Omega_2$

- Marginal distributions:

Are the respective distributions $\mathbb{P}_{X_1}$ and $\mathbb{P}_{X_2}$ derived from the probabilities $\mathbb{P}_{X_1,X_2}$.

$$\{X_1 = x\} = \bigcup_{x_i \in \Omega_2} \{X_1 = x; X_2 = x_i\}$$

$$\mathbb{P}_{X_1}(X_1 = x) = \mathbb{P}\Big(\bigcup_{x_i \in \Omega_2} \{X_1 = x; X_2 = x_i\}\Big) = \sum_{x_i \in \Omega_2} \mathbb{P}_{X_1,X_2}(X_1 = x; X_2 = x_i)$$

$$= \sum_{x_i \in \Omega_2} \mathbb{P}_{X_1,X_2}(x, x_i)$$

- In the same way

$$\mathbb{P}_{x_2}(X_2 = x) = \sum_{x_i \in \Omega_1} \mathbb{P}_{X_1,X_2}(x_i, x)$$

# Multivariate Random Variables

- For the previous example: $\Omega_1 \times \Omega_2 = \{(HHH, HHH), (HHH.HHT), \ldots\} \rightarrow 64$ elements

| $x_i / y_i$ | 0 | 1 | 2 | 3 | $\mathbb{P}_X = \sum_{y_i} \mathbb{P}(X; Y = y_i)$ |
|---|---|---|---|---|---|
| 0 | $\frac{1}{64}$ | $\frac{3}{64}$ | $\frac{3}{64}$ | $\frac{1}{64}$ | $\frac{8}{64}$ |
| 1 | $\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}$ | $\frac{24}{64}$ |
| 2 | $\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}$ | $\frac{24}{64}$ |
| 3 | $\frac{1}{64}$ | $\frac{3}{64}$ | $\frac{3}{64}$ | $\frac{1}{64}$ | $\frac{8}{64}$ |
| $\mathbb{P}_Y = \sum_{x_i} \mathbb{P}(X = x_i; Y)$ | $\frac{8}{64}$ | $\frac{24}{64}$ | $\frac{24}{64}$ | $\frac{8}{64}$ | $\sum_i \mathbb{P}_{X,Y} = 1$ |

$\rightarrow$ Marginal $\mathbb{P}_X$

$\downarrow$ Marginal $\mathbb{P}_Y$

$$\mathbb{P}_X(X = 0) = \sum_{all \ \ y_i} \mathbb{P}(X = 0, Y = y_i) = \mathbb{P}(X = 0, Y = 0)$$

$$+\mathbb{P}(X = 0, Y = 1) + \mathbb{P}(X = 0, Y = 2) + \mathbb{P}(X = 0, Y = 3) = \frac{1}{64} + \frac{3}{64} + \frac{3}{64} + \frac{1}{64} = \frac{8}{64}$$

$$\mathbb{P}_X(X = 2) = \sum_{y_i \in \Omega_2} \mathbb{P}(X = 2, Y = y_i) = \mathbb{P}(X = 2, Y = 0)$$

$$+\mathbb{P}(X = 2, Y = 1) + \mathbb{P}(X = 2, Y = 2) + \mathbb{P}(X = 2, Y = 3) = \frac{3}{64} + \frac{9}{64} + \frac{9}{64} + \frac{3}{64} = \frac{24}{64}$$

# Multivariate Random Variables

- Illustrative example 2:

1) Let $\Omega =$ "The number of people moving to Knoxville". A hypothetical statistical analysis shows that 50% move to attend UT and 50% for professional position. If you ask 3 new arrivals, so we have

$X_1 =$"# of person who come to attend UT" $\rightarrow X_1 \rightarrow \mathcal{B}(n,p) = \mathcal{B}(3, 0.5)$.

$$\Rightarrow \mathbb{P}(X_1 = k) = C_k^3 (0.5)^k (1-0.5)^{3-k}$$

$X_2 =$"# of person who come for a professional position" $\rightarrow X_2 \rightarrow \mathcal{B}(n,p) = \mathcal{B}(3, 0.5)$.

$$\Rightarrow \mathbb{P}(X_2 = k) = C_k^3 (0.5)^k (1-0.5)^{3-k}$$

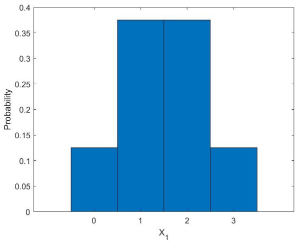
| $x_i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mathbb{P}_{X_1}$ | 0.125 | 0.375 | 0.375 | 0.1253 |

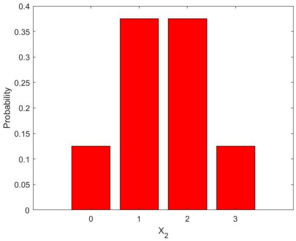
$=$

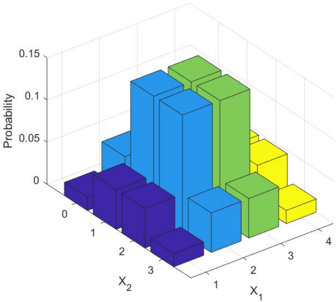
| $x_i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mathbb{P}_{X_2}$ | 0.125 | 0.375 | 0.375 | 0.1253 |

| $X_1 \| X_2$ | 0 | 1 | 2 | 3 | $\sum_{x_i \in X_2} x_i$ |
|---|---|---|---|---|---|
| 0 | 0.0156 | 0.0469 | 0.0469 | 0.0156 | 0.125 |
| 1 | 0.0469 | 0.1406 | 0.1406 | 0.0469 | 0.375 |
| 2 | 0.0469 | 0.1406 | 0.1406 | 0.0469 | 0.375 |
| 3 | 0.0156 | 0.0469 | 0.0469 | 0.0156 | 0.125 |
| $\sum_{x_i \in X_1}$ | 0.0156 | 0.0469 | 0.0469 | 0.0156 | $\sum_{X_1, X_2} = 1$ |

# Multivariate Random Variables



# Multivariate Random Variables

- Illustrative example 3:

1) Let $\Omega =$ "The number of people moving to Knoxville". A hypothetical statistical analysis shows that 25% move to attend UT, 55% for professional position and 20% for other reasons. If you ask 10 new arrivals, so we have

$X_1 =$"# of person who come to attend UT" $\rightarrow X_1 \rightarrow \mathcal{B}(n,p) = \mathcal{B}(10, 0.25)$.

$$\Rightarrow \mathbb{P}(X_1 = k) = C_k^{10}(0.25)^k(1-0.25)^{10-k}$$

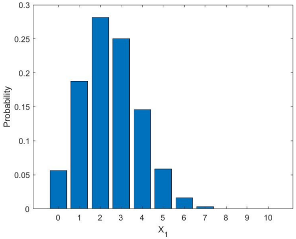
$X_2 =$"# of person who come for a professional position" $\rightarrow X_2 \rightarrow \mathcal{B}(n,p) = \mathcal{B}(10, 0.55)$.

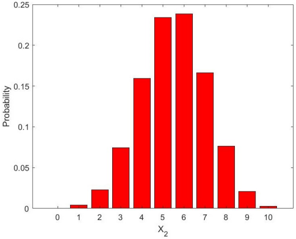
$$\Rightarrow \mathbb{P}(X_2 = k) = C_k^{10}(0.55)^k(1-0.55)^{10-k}$$

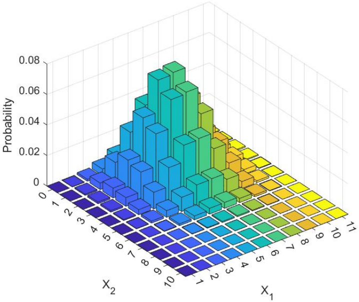
| $x_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{P}_{X_1}$ | 0.0563 | 0.1877 | 0.2816 | 0.2503 | 0.146 | 0.0584 | 0.0162 | 0.0031 | 0.0004 | 0 | 0 |

| $x_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{P}_{X_2}$ | 0.0003 | 0.0042 | 0.0229 | 0.0746 | 0.1596 | 0.234 | 0.2384 | 0.1665 | 0.0763 | 0.0207 | 0.0025 |

# Multivariate Random Variables



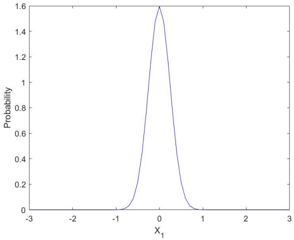



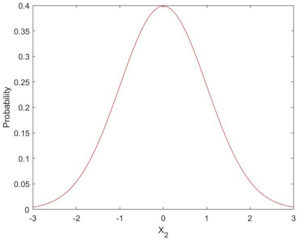



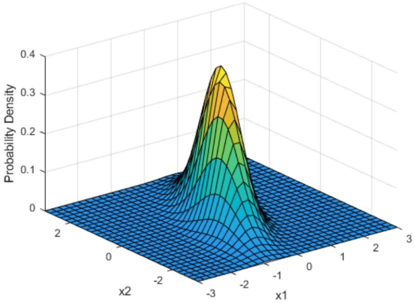
# Multivariate Random Variables

- Illustrative example 4:

$X_1 \to \mathcal{N}(0, 0.25)$ and $X_2 \to \mathcal{N}(0, 1)$



# Multivariate Random Variables

## - Conditional Probability:

Gives us information about how the knowledge of one random variable's outcome may affect the other. We formalize this as a conditional probability function, defined by $\mathbb{P}(Y = y|X = x) = \mathbb{P}_{Y,X}(y|x)$ which we read: "the probability of $Y = y$ given that (or knowing that) $X = x$."

$$\mathbb{P}(Y = y|X = x) = \frac{\mathbb{P}(Y = y \cap X = x)}{\mathbb{P}(X = x)} = \frac{\mathbb{P}_{X,Y}(x, y)}{\mathbb{P}_X(x)}$$

- If $\mathbb{P}_{Y,X}(y|x) = \mathbb{P}_Y(y)$ for all possible pairs of values $(x, y)$, we say that $X$ and $Y$ are independent.
- If $X$ and $Y$ are independent, we have

$$\mathbb{P}(y|x) = \frac{\mathbb{P}(Y = y \cap X = x)}{\mathbb{P}(X = x)} = \frac{\mathbb{P}_{X,Y}(x, y)}{\mathbb{P}_X(x)} = \mathbb{P}_Y(y) \Rightarrow \mathbb{P}_{X,Y}(x, y) = \mathbb{P}_X(x)\mathbb{P}_Y(y)$$

# Multivariate Random Variables

- Expectation of bivariant random variables: Let the bivarainte random variable $(X, Y)$.

1) We can consider the expectation $\mathbb{E}(X, Y) = (\mathbb{E}(X), \mathbb{E}(Y))$

2) Let $h(X, Y)$ be any function of $(X, Y)$. For example:

$$h(X, Y) = X - Y, \quad h(X, Y) = 2X + Y, \quad h(X, Y) = XY$$

Then, the expectation of $h(X, Y)$ is given by

$$\mathbb{E}(h(X, Y)) = \sum \sum\nolimits_{(x,y) \in X \times Y} h(x, y)\ \mathbb{P}_{X,Y}(x, y)$$

- If $h(X, Y) = X$, we get

$$\mu_X = \mathbb{E}(X) = \sum_{x \in S_1} \sum_{y \in S_2} x\ \mathbb{P}_{X,Y}(x, y)$$

- If $h(X, Y) = Y$, we get $\mu_Y = \mathbb{E}(Y) = \sum_{x \in S_1} \sum_{y \in S_2} y\ \mathbb{P}_{X,Y}(x, y)$

# Multivariate Random Variables

Example:

- If $h(X,Y) = X + Y$, we get

$$\mathbb{E}(X+Y) = \sum_{x_i}\sum_{y_i}(x_i + y_i)\ \mathbb{P}_{X,Y}(x_i,y_i) = \sum_{x_i}\sum_{y_i}[x_i\mathbb{P}_{X,Y}(x_i,y_i) + y_i\ \mathbb{P}_{X,Y}(x_i,y_i)]$$

$$= [\sum_{x_i} x_i\mathbb{P}_{X,Y}(x_i,y_i)] + [\sum_{y_i} y_i\ \mathbb{P}_{X,Y}(x_i,y_i)] = \mathbb{E}(X) + \mathbb{E}(Y)$$

- If $h(X,Y) = XY$, we get

$$\mathbb{E}(XY) = \sum_{x_i}\sum_{y_i} x_i\ y_i\ \mathbb{P}_{X,Y}(x_i,y_i)$$

- If $h(X,Y) = 2X - Y$, we get

$$\mathbb{E}(2X-Y) = \sum_{x_i}\sum_{y_i}(2x_i - y_i)\ \mathbb{P}_{X,Y}(x_i,y_i) = \sum_{x_i}\sum_{y_i}[2x_i\mathbb{P}_{X,Y}(x_i,y_i) - y_i\ \mathbb{P}_{X,Y}(x_i,y_i)]$$

$$= 2\mathbb{E}(X) - \mathbb{E}(Y)$$

# Multivariate Random Variables

**Problem:** Consider the following joint PMF of the random variables $X$ and $Y$

| $X \mid Y$ | 1 | 2 | 3 | 4 | $\mathbb{P}_X(x)$ |
|---|---|---|---|---|---|
| 1 | 0 | 0 | $\frac{2}{20}$ | $\frac{1}{20}$ | |
| 2 | $\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{4}{20}$ | $\frac{2}{20}$ | |
| 3 | 0 | $\frac{3}{20}$ | $\frac{1}{20}$ | $\frac{2}{20}$ | |
| 4 | 0 | $\frac{1}{20}$ | $\frac{2}{20}$ | 0 | |
| $\mathbb{P}_Y(y)$ | | | | | |

1) Compute the marginal probabilities
2) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[XY]$, $\mathbb{E}[3X - 2Y]$ and $\mathbb{E}[3X - 2Y + 1]$.
3) Compute the conditional probabilities : $\mathbb{P}(X = x_i | Y = y_i)$ for all $x_i$ and $y_i$
4) Are $X$ and $Y$ independent?
5) Compute $\mathbb{P}(x \leq 2 | Y \geq 3)$
6) Compute $\mathbb{P}(X + Y = 4)$

# Multivariate Random Variables

**Solution:**

1) Compute the marginal probabilities

| $X \mid Y$ | 1 | 2 | 3 | 4 | $\mathbb{P}_X(x)$ |
|---|---|---|---|---|---|
| 1 | 0 | 0 | $\frac{2}{20}$ | $\frac{1}{20}$ | $\frac{3}{20}$ |
| 2 | $\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{4}{20}$ | $\frac{2}{20}$ | $\frac{8}{20}$ |
| 3 | 0 | $\frac{3}{20}$ | $\frac{1}{20}$ | $\frac{2}{20}$ | $\frac{6}{20}$ |
| 4 | 0 | $\frac{1}{20}$ | $\frac{2}{20}$ | 0 | $\frac{3}{20}$ |
| $\mathbb{P}_Y(y)$ | $\frac{1}{20}$ | $\frac{5}{20}$ | $\frac{9}{20}$ | $\frac{5}{20}$ | 1 |

2) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[XY]$, $\mathbb{E}[3X-2Y]$ and $\mathbb{E}[3X-2Y+1]$

- $\mathbb{E}[X] = \sum_i x_i \mathbb{P}(X = x_i) = (1 \times \frac{3}{20}) + (2 \times \frac{8}{20}) + (3 \times \frac{6}{20}) + (4 \times \frac{3}{20}) = \frac{49}{20} = 2.45$
- $\mathbb{E}[Y] = \sum_i y_i \mathbb{P}(Y = y_i) = (1 \times \frac{1}{20}) + (2 \times \frac{5}{20}) + (3 \times \frac{9}{20}) + (4 \times \frac{5}{20}) = \frac{58}{20} = 2.9$

# Multivariate Random Variables

| $XY$ | 1 | 2 | 3 | 4 | 6 | 8 | 9 | 12 | 16 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{P}(XY)$ | 0 | $\frac{1}{20}+0$ | $\frac{2}{20}+0$ | $\frac{1}{20}+0+\frac{1}{20}$ | $\frac{4}{20}+\frac{3}{20}$ | $\frac{2}{20}+\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{2}{20}+\frac{2}{20}$ | 0 |

| $XY$ | 1 | 2 | 3 | 4 | 6 | 8 | 9 | 12 | 16 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{P}(XY)$ | 0 | $\frac{1}{20}$ | $\frac{2}{20}$ | $\frac{2}{20}$ | $\frac{7}{20}$ | $\frac{3}{20}$ | $\frac{1}{20}$ | $\frac{4}{20}$ | 0 |

- $\mathbb{E}[XY] = (1\times 0)+(2\times\frac{1}{20})+(3\times\frac{2}{20})+(4\times\frac{2}{20})+(6\times\frac{7}{20})+(8\times\frac{3}{20})+(9\times\frac{1}{20})+(12\times\frac{4}{20})+(16\times 0) = 6.95$

- $\mathbb{E}[3X-2Y] = 3\mathbb{E}[X]-2\mathbb{E}[Y] = 3(2.45)-2(2.9) = 1.55$
- $\mathbb{E}[3X-2Y+1] = 3\mathbb{E}[X]-2\mathbb{E}[Y]+1 = 3(2.45)-2(2.9)+1 = 2.55$

# Multivariate Random Variables

3) Compute the conditional probabilities :

$$\mathbb{P}(X = x_i | Y = y_i) = \frac{\mathbb{P}(X = x_i; Y = y_i)}{\mathbb{P}(Y = y_i)} \rightarrow \mathbb{P}(X = 2 | Y = 3) = \frac{\mathbb{P}(X = 2; Y = 3)}{\mathbb{P}(Y = 3)}$$

- We can summarize all the conditional probabilities in the following table

| $X\|Y$ | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| 1 | $\frac{0}{\frac{1}{20}} = 0$ | $\frac{0}{\frac{5}{20}} = 0$ | $\frac{\frac{2}{20}}{\frac{9}{20}} = \frac{2}{9}$ | $\frac{\frac{1}{20}}{\frac{5}{20}} = \frac{1}{5}$ | $\mathbb{P}(X = 1 \| Y = y_i)$ |
| 2 | $\frac{\frac{1}{20}}{\frac{1}{20}} = 1$ | $\frac{\frac{1}{20}}{\frac{5}{20}} = \frac{1}{5}$ | $\frac{\frac{4}{20}}{\frac{9}{20}} = \frac{4}{9}$ | $\frac{\frac{2}{20}}{\frac{5}{20}} = \frac{2}{5}$ | $\mathbb{P}(X = 2 \| Y = y_i)$ |
| 3 | $\frac{0}{\frac{1}{20}} = 0$ | $\frac{\frac{3}{20}}{\frac{5}{20}} = \frac{3}{5}$ | $\frac{\frac{1}{20}}{\frac{9}{20}} = \frac{1}{9}$ | $\frac{\frac{2}{20}}{\frac{5}{20}} = \frac{2}{5}$ | $\mathbb{P}(X = 3 \| Y = y_i)$ |
| 4 | $\frac{0}{\frac{1}{20}} = 0$ | $\frac{\frac{1}{20}}{\frac{5}{20}} = \frac{1}{5}$ | $\frac{\frac{2}{20}}{\frac{9}{20}} = \frac{2}{9}$ | $\frac{0}{\frac{5}{20}} = 0$ | $\mathbb{P}(X = 4 \| Y = y_i)$ |
| $\sum \mathbb{P}_{X,Y}(X = x_i \| Y = y_i)$ | 1 | 1 | 1 | 1 | |

▷ Remark: We can check the independence of two random variables using the conditional probability

if $\mathbb{P}(X = x_i | Y = y_i) = \mathbb{P}(X = x_i)$, for all $x_i$ and $y_i \Rightarrow$ X and Y are independent

OR

if $\mathbb{P}(X = x_i; Y = y_i) = \mathbb{P}(X = x_i).\mathbb{P}(Y = y_i)$, for all $x_i$ and $y_i \Rightarrow$ X and Y are independent

# Multivariate Random Variables

- We can remark that

$$\mathbb{P}(X = 1 | Y = 1) = 0 \neq \mathbb{P}(X = 1) = \frac{3}{20}$$

- Also we can remark that

$$\mathbb{P}(X = 1 \cap Y = 1) = 0 \neq \mathbb{P}(X = 1).\mathbb{P}(Y = 1) = \frac{3}{20}.\frac{1}{20} = \frac{3}{400}$$

$\rightarrow$ So X and Y are not independents.

Rule: If X and Y are independent $\Rightarrow \mathbb{E}[XY] = \mathbb{E}[X].\mathbb{E}[Y]$ $\Leftrightarrow$ $\mathbb{E}[XY] \neq \mathbb{E}[X].\mathbb{E}[Y] \Rightarrow$ X and Y are not independent.

**Proof:**

$$\mathbb{E}[XY] = \sum_{x_i}\sum_{y_i} x_i y_i \mathbb{P}(X = x_i; Y = y_i) \underbrace{\longrightarrow}_{\text{if X and Y are independents}} \mathbb{E}[XY] = \sum_{x_i}\sum_{y_i} x_i y_i \mathbb{P}(X = x_i).\mathbb{P}(Y = y_i)$$

$$\Rightarrow \mathbb{E}[XY] = \sum_{x_i} x_i \mathbb{P}(X = x_i) \sum_{y_i} y_i \mathbb{P}(Y = y_i) = \mathbb{E}[X].\mathbb{E}[Y]$$

- In our problem $\rightarrow \mathbb{E}[X].\mathbb{E}[Y] = 2.45 \times 2.9 = 7.1050 \neq 6.95 \Rightarrow$ X and Y are not independent (or dependent)

# Multivariate Random Variables

Counter example: If $\mathbb{E}[XY] = \mathbb{E}[X].\mathbb{E}[Y] \nRightarrow X$ and $Y$ are independent

| $X\|Y$ | -1 | 0 | 1 | $\mathbb{P}(X)$ |
|---|---|---|---|---|
| -1 | 0 | $\frac{1}{4}$ | 0 | $\mathbb{P}(X=-1)=\frac{1}{4}$ |
| 0 | $\frac{1}{4}$ | 0 | $\frac{1}{4}$ | $\mathbb{P}(X=0)=\frac{1}{2}$ |
| 1 | 0 | $\frac{1}{4}$ | 0 | $\mathbb{P}(X=1)=\frac{1}{4}$ |
| $\mathbb{P}(Y)$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | 1 |

- $\mathbb{E}[X] = \sum_{x_i} x_i.\mathbb{P}(X = x_i) = (-1)(\frac{1}{4}) + 0.\frac{1}{2} + 1.\frac{1}{4} = 0$
- $\mathbb{E}[Y] = \sum_{y_i} y_i.\mathbb{P}(Y = y_i) = (-1)(\frac{1}{4}) + 0.\frac{1}{2} + 1.\frac{1}{4} = 0$

| $XY$ | -1 | 0 | 1 |
|---|---|---|---|
| $\mathbb{P}(X;Y)$ | 0+0=0 | $\frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$ | 0+0 =0 |

- $\mathbb{E}[XY] = \sum_i x_i y_i.\mathbb{P}(X = x_i; Y = y_i) = (-1)(0) + 0.(1) + 1(0) = 0$
- We can remark that $\mathbb{E}[XY] = \mathbb{E}[X].\mathbb{E}[Y] = 0$ but

$\mathbb{P}(X = -1; Y = -1) = 0 \neq \mathbb{P}(X = -1).\mathbb{P}(Y = -1) = \dfrac{1}{4}.\dfrac{1}{4} \Rightarrow$ X and Y are not independent.

# Multivariate Random Variables

5) Compute $\mathbb{P}(X \leq 2 | Y \geq 3) = \dfrac{\mathbb{P}(X \leq 2; Y \geq 3)}{\mathbb{P}(Y \geq 3)}$

- $\mathbb{P}(X \leq 2; Y \geq 3) = \mathbb{P}(X = \{1,2\}; Y = \{3,4\})$

$$\mathbb{P}(X=1; Y=3) + \mathbb{P}(X=2; Y=3) + \mathbb{P}(X=1; Y=4) + \mathbb{P}(X=2; Y=4) = \frac{9}{20}$$

- $\mathbb{P}(Y \geq 3) = \mathbb{P}(Y=3) + \mathbb{P}(Y=4) = \dfrac{9}{20} + \dfrac{5}{20} = \dfrac{14}{20}$

| $X/Y$ | 1 | 2 | 3 | 4 | $\mathbb{P}_X(x)$ |
|---|---|---|---|---|---|
| 1 | 0 | 0 | $\frac{2}{20}$ | $\frac{1}{20}$ | $\frac{3}{20}$ |
| 2 | $\frac{1}{20}$ | $\frac{1}{20}$ | $\frac{4}{20}$ | $\frac{2}{20}$ | $\frac{8}{20}$ |
| 3 | 0 | $\frac{3}{20}$ | $\frac{1}{20}$ | $\frac{2}{20}$ | $\frac{6}{20}$ |
| 4 | 0 | $\frac{1}{20}$ | $\frac{2}{20}$ | 0 | $\frac{3}{20}$ |
| $\mathbb{P}_Y(y)$ | $\frac{1}{20}$ | $\frac{5}{20}$ | $\frac{9}{20}$ | $\frac{5}{20}$ | 1 |

$$\mathbb{P}(X \leq 2 | Y \geq 3) = \frac{\mathbb{P}(X \leq 2; Y \geq 3)}{\mathbb{P}(Y \geq 3)} = \frac{\frac{9}{20}}{\frac{14}{20}} = \frac{9}{14}$$

# Multivariate Random Variables

6) Compute $\mathbb{P}(X + Y = 4)$

- Let $X + Y = 4 \Rightarrow Y = 4 - X \Rightarrow \begin{cases} X = 1 & \Rightarrow & Y = 3 & \rightarrow & \mathbb{P}(X = 1; Y = 3) = \dfrac{2}{20} \\ X = 2 & \Rightarrow & Y = 2 & \rightarrow & \mathbb{P}(X = 2; Y = 2) = \dfrac{1}{20} \\ X = 3 & \Rightarrow & Y = 1 & \rightarrow & \mathbb{P}(X = 3; Y = 1) = 0 \\ X = 4 & \Rightarrow & Y = 0 & \rightarrow & \text{Impossible event} \end{cases}$

$$\mathbb{P}(X + Y = 4) = \frac{2}{20} + \frac{1}{20} + 0 = \frac{3}{20}$$

# Multivariate Random Variables

Covariance: Covariance is a measure of how much two random variables vary together according to the spread of the random variables around their means. Note that, the variance tells you how a single variable varies around the expectation, covariance tells you how two variables vary together.

- Let $X$ and $Y$ be random variables with means $\mu_X$ and $\mu_Y$. The covariance of $X$ and $Y$, denoted by $Cov(X,Y)$ is defined as:

$$Cov(X,Y) = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] = \sum_{(x,y)} (x-\mu_X)(y-\mu_Y)\mathbb{P}_{X,Y}(x,y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

- **Proof:**

$$\sum_{(x,y)} (x-\mu_X)(y-\mu_Y)\mathbb{P}_{X,Y}(x,y) = \sum_{(x,y)} [xy - \mu_X y - y\mu_Y + \mu_X\mu_Y]\mathbb{P}_{X,Y}(x,y)$$

$$\underbrace{\sum_{(x,y)} xy\mathbb{P}_{X,Y}(x,y)}_{\mathbb{E}(XY)} - \underbrace{\mu_X \sum_{(x,y)} y\mathbb{P}_{X,Y}(x,y)}_{\mu_X\mathbb{E}(Y)=\mathbb{E}(X)\mathbb{E}(Y)} - \underbrace{\mu_Y \sum_{(x,y)} x\mathbb{P}_{X,Y}(x,y)}_{\mu_Y\mathbb{E}(X)=\mathbb{E}(Y)\mathbb{E}(X)} + \underbrace{\mu_X\mu_Y}_{=\mathbb{E}(X)\mathbb{E}(Y)} \underbrace{\sum_{(x,y)} \mathbb{P}_{X,Y}(x,y)}_{=1}$$

$$= \mathbb{E}(XY) - 2\mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

- $Cov(X,X) = \mathbb{E}[XX] - \mathbb{E}(X)\mathbb{E}(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X] = V(X)$.

# Multivariate Random Variables

- Example: Suppose that $X$ and $Y$ have the following joint PMF

| X—Y | 1 | 2 | 3 | $\mathbb{P}(X)$ |
|---|---|---|---|---|
| 1 | 0.25 | 0.25 | 0 | $\mathbb{P}(X=1)=0.5$ |
| 2 | 0 | 0.25 | 0.25 | $\mathbb{P}(X=2)=0.5$ |
| $\mathbb{P}(Y)$ | $\mathbb{P}(Y=1)=0.25$ | $\mathbb{P}(Y=2)=0.5$ | $\mathbb{P}(Y=3)=0.25$ | 1 |

We have

$$\mu_X = \mathbb{E}(X) = \sum_{(x,y)} x\mathbb{P}_{X,Y}(x,y) = \sum_{y_i} 1\ \mathbb{P}_{X,Y}(1,y_i) + \sum_{y_i} 2\ \mathbb{P}_{X,Y}(2,y_i) = \sum_{x} x\mathbb{P}_X(x)$$

$$= 1(0.5) + 2(0.5) = 1.5$$

$$\mu_Y = \mathbb{E}(Y) = \sum_{(x,y)} y\mathbb{P}_{X,Y}(x,y) = \sum_{x_i} 1\ \mathbb{P}_{X,Y}(x_i,1) + \sum_{x_i} 2\ \mathbb{P}_{X,Y}(x_i,2) + \sum_{x_i} 3\mathbb{P}_{X,Y}(x_i,3)$$

$$= \sum_{Y} y\mathbb{P}_Y(y) = 1(0.25) + 2(0.5) + 3(0.25) = 2$$

$\mu_X = 1.5$ and $\mu_Y = 2$. What is the covariance of $X$ and $Y$?

# Multivariate Random Variables

- Solution: 1) **Method-1:** $Cov(X,Y)=\sum_{(x,y)}(x-\mu_X)(y-\mu_Y)\mathbb{P}_{X,Y}(x,y)$

| $X-\mu_X \mid Y-\mu_Y$ | 1-2=-1 | 2-2=0 | 3-2=1 |
|---|---|---|---|
| 1-1.5 = -0.5 | 0.25 | 0.25 | 0 |
| 2-1.5 = 0.5 | 0 | 0.25 | 0.25 |

| $(x_i-\mu_X)(y_i-\mu_Y)$ | $-1\times-0.5$ | $-1\times0.5$ | $0\times-0.5$ | $0\times0.5$ | $1\times-0.5$ | $1\times0.5$ |
|---|---|---|---|---|---|---|
| $\mathbb{P}(X=x_i;Y=y_i)$ | 0.25 | 0 | 0.25 | 0.25 | 0 | 0.25 |

| $(x_i-\mu_X)(y_i-\mu_Y)$ | 0.5 | $-0.5$ | 0 | 0 | $-0.5$ | 0.5 |
|---|---|---|---|---|---|---|
| $\mathbb{P}(X=x_i;Y=y_i)$ | 0.25 | 0 | 0.25 | 0.25 | 0 | 0.25 |

$Cov(X,Y)=\sum_{(x,y)}(x-\mu_X)(y-\mu_Y)\mathbb{P}_{X,Y}(x,y)=$
$(0.5\times0.25)+(-0.5\times0)+(0\times0.25)+(0\times0.25)+(-0.5\times0)+(0.5\times0.25)=0.25$

# Multivariate Random Variables

2) **Method-2:** $Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

| XY | 1 | 2 | 3 | 4 | 6 |
|---|---|---|---|---|---|
| $\mathbb{P}(XY)$ | 0.25 | 0.25 + 0 | 0 | 0.25 | 0.25 |

- $XY = 1 \Rightarrow X = 1 \text{ and } Y = 1 \Rightarrow \mathbb{P}(XY = 1) = \mathbb{P}(X = 1; Y = 1) = 0.25$
- $XY = 2 \Rightarrow (X = 1 \text{ and } Y = 2) \text{ Or } (X = 2 \text{ and } Y = 1)$
  $\Rightarrow \mathbb{P}(XY = 2) = \mathbb{P}(X = 1; Y = 2) + \mathbb{P}(X = 2; Y = 1) = 0.25 + 0 = 0.25$
- $XY = 3 \Rightarrow X = 1 \text{ and } Y = 3 \Rightarrow \mathbb{P}(XY = 3) = \mathbb{P}(X = 1; Y = 3) = 0$
- $XY = 4 \Rightarrow X = 2 \text{ and } Y = 2 \Rightarrow \mathbb{P}(XY = 4) = \mathbb{P}(X = 2; Y = 2) = 0.25$
- $XY = 6 \Rightarrow X = 2 \text{ and } Y = 3 \Rightarrow \mathbb{P}(XY = 6) = \mathbb{P}(X = 2; Y = 3) = 0.25$

$$\mathbb{E}(XY) = \sum x_i y_i \, \mathbb{P}_{X,Y}(x_i, y_i) = (1 \times 0.25) + (2 \times 0.25) + (3 \times 0) + (4 \times 0.25) + (6 \times 0.25) = \frac{13}{4}$$

$$Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \frac{13}{4} - (1.5 \times 2) = \frac{1}{4} = 0.25.$$

# Multivariate Random Variables

- $V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$

Proof:

$$V(X+Y) = \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 = \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2$$

$$= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}^2[X] - 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}^2[Y]$$

$$= \underbrace{(\mathbb{E}[X^2] - \mathbb{E}^2[X])}_{V(X)} + \underbrace{(\mathbb{E}[Y^2] - \mathbb{E}^2[Y])}_{V(Y)} + 2\underbrace{(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])}_{Cov(X,Y)}$$

$$\Rightarrow V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$$

- We have also the following property:

$Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$

# Multivariate Random Variables

-Interpretation:

▷ Covariance measures the total variation of two random variables from their means.

▷ Using covariance we can only gauge the relationship between two random variables (whether the variables tend to move in tandem or show an inverse relationship). However, it does not indicate the strength of the relationship.

▷ Covariance is measured in units:

-If $X$ is the size of houses in $m^2$ and $Y$ is the price in \$

$\rightarrow Cov(X,Y) = \sum_i (x_i - \mu_X)(y_i - \mu_i)\mathbb{P}_{X,Y}(x_i, y_i)$ is in $m^2 \times m^2 \times \$$

▷ Covariance can take any positive or negative values ($-\infty < Cov(X,Y) < +\infty$):

- **Positive covariance:** Indicates that two variables tend to move in the same direction.
- **Negative covariance:** Reveals that two variables tend to move in inverse directions.

- A large covariance can mean a strong relationship between variables

$\rightarrow$ The main problem with interpretation is that the wide range of results that it takes on makes it hard to interpret $\rightarrow$ The larger the $X$ and $Y$ values, the larger the covariance.

$$\text{- If } Cov(X,Y) = 123\ m^2 \times \$ \Rightarrow Cov(X,Y) = 12300\ cm^2 \times \$$$

$\rightarrow$ The problem can be fixed by dividing the covariance by the standard deviation to get the correlation coefficient.

# Multivariate Random Variables

- Derivation and properties of the Correlation Coefficient:

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}}$$

• Cauchy-Schwartz inequality: Let the two vectors $X = (x_i)_{i\in\mathbb{N}}$ and $Y = (y_i)_{i\in\mathbb{N}}$

$$|X.Y| \leq \|X\|.\|Y\| \rightarrow |x_1y_1+x_2y_2+\ldots+x_ny_n| \leq \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.\sqrt{y_1^2 + y_2^2 + \ldots + y_n^2}$$

$$\Rightarrow (\sum_i^n x_iy_i)^2 \leq (\sum_i^n x_i^2)(\sum_i^n y_i)^2$$

$$Cov(X, Y) = \sum_i (x_i - \mu_X)(y_i - \mu_i)\mathbb{P}(X, Y) \Rightarrow Cov^2(X, Y) = [\sum_i (x_i - \mu_X)(y_i - \mu_i)\mathbb{P}_{(X,Y)}]^2$$

$$\leq [\sum_i (x_i-\mu_X)^2].[\sum_i (y_i-\mu_i)^2]\mathbb{P}^2_{(X,Y)} = [\sum_i (x_i-\mu_X)^2\mathbb{P}_{(X,Y)}].[\sum_i (y_i-\mu_i)^2\mathbb{P}_{(X,Y)}] = V(X).V(Y)$$

$$\Rightarrow -\sqrt{V(X)V(Y)} \leq Cov(X, Y) \leq \sqrt{V(X)V(Y)} \Rightarrow -1 \leq Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}} \leq +1$$

# Multivariate Random Variables

- Advantages of the Correlation Coefficient:

1) Correlation coefficient is limited between 1 and -1: $-1 \leq Corr(X, Y) \leq +1$.
2) Because of it's numerical limitations, correlation is more useful for determining how strong the relationship is between the two variables.
3) Correlation does not have a unit $\rightarrow \dfrac{Cov(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}} \rightarrow \dfrac{m^2 \times \$}{m^2 \times \$}$
4) Correlation isn't affected by changes in the center (i.e. mean) or scale of the variables

- Correlation Coefficient:

• When $Corr(X, Y) = 1$, the random variables have a perfect positive correlation (when one moves, so does the other in the same direction, proportionally).

• When $0 < Corr(X, Y) < 1$, it indicates a less than perfect positive correlation $\rightarrow$ The closer the correlation coefficient gets to one, the stronger the correlation between the two variables.

• When $Corr(X, Y) = 0$, it means that there is no identifiable relationship between the variables (if one variable moves, it's impossible to make predictions about the movement of the other variable.

# Multivariate Random Variables

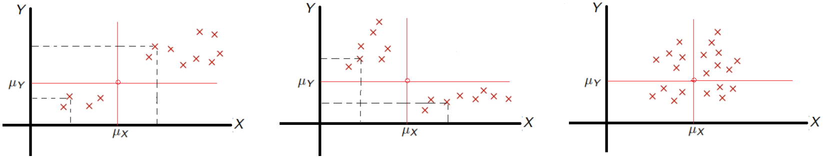
- When $-1 < Corr(X, Y) < 0 \rightarrow$ The variables are negatively or inversely correlated.
- When $Corr(X, Y) = -1 \rightarrow$ The variables are perfectly negatively or inversely correlated

(The variables will move in opposite directions from each other) $\Rightarrow$ If one variable increases, the other will decrease at the same proportion.

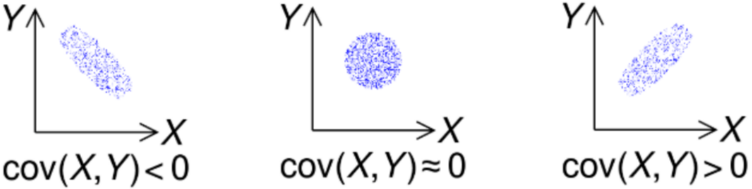
- How the correlation get its sign

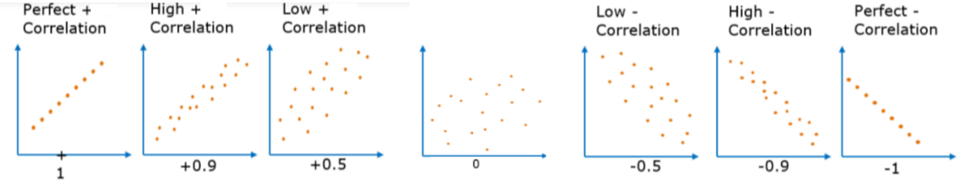
$$Corr(X,Y) = \frac{Cov(x,Y)}{\sigma_X \sigma_Y} = \frac{\sum_{(x,y)} \underbrace{(x-\mu_X)(y-\mu_Y)}_{<0 \text{ Or } >0} \underbrace{\mathbb{P}_{X,Y}(x,y)}_{>0}}{\underbrace{\sigma_X \sigma_Y}_{>0}}$$

- The sign of the $Corr(X, Y)$ depend on the sign of the product $(x - \mu_X)(y - \mu_Y)$ which represent the spread of the values of X and Y around their means.



# Multivariate Random Variables



# Multivariate Random Variables

- Example: Consider the previous example where $X$ and $Y$ have the following PMF

| $\mathbb{P}(X, Y)$ | | 1 | 2 | 3 | $p_X(x)$ |
|---|---|---|---|---|---|
| $x$ | 1 | 0.25 | 0.25 | 0 | 0.5 |
| | 2 | 0 | 0.25 | 0.25 | 0.5 |
| $p_Y(y)$ | | 0.25 | 0.5 | 0.25 | 1 |

We computed $\mu_X = 1.5$, $\mu_Y = 2$ and $Cov(X, Y) = 0.25$

$$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = [\sum_X x_i^2 \mathbb{P}_X(x_i)] - \mu_X^2 = [(1^2 \times 0.5) + (2^2 \times 0.5)] - (1.5)^2 = 0.25$$

$$V(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = [\sum_Y y_i^2 \mathbb{P}_Y(y_i)] - \mu_Y^2 = [(1^2 \times 0.25) + (2^2 \times 0.5) + (3^2 \times 0.25)] - 2^2 = 0.5$$

- Let us compute the correlation coefficient

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{Cov(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{0.25}{\sqrt{0.25 \times 0.5}} = 0.707$$

$\rightarrow$ X and Y are positively correlated.

# Multivariate Random Variables

- Example: Consider the example of tossing two different coins 3 times

| | $x_2$ | | | | |
|---|---|---|---|---|---|
| $x_1$ | 0 | 1 | 2 | 3 | $P(x_1)$ |
| 0 | $\frac{1}{64}$ | $\frac{3}{64}$ | $\frac{3}{64}$ | $\frac{1}{64}$ | $\frac{1}{8}$ |
| 1 | $\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}$ | $\frac{3}{8}$ |
| 2 | $\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}$ | $\frac{3}{8}$ |
| 3 | $\frac{1}{64}$ | $\frac{3}{64}$ | $\frac{3}{64}$ | $\frac{1}{64}$ | $\frac{1}{8}$ |
| $P(x_2)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ | |

$$\mu_{X_1} = \mathbb{E}(X_1) = \sum_{X_1} x_i \mathbb{P}_{X_1}(x_i) = (0 \times \frac{1}{8}) + (1 \times \frac{3}{8}) + (2 \times \frac{3}{8}) + (3 \times \frac{1}{8}) = 1.5$$
$$\mu_{X_2} = \mathbb{E}(X_2) = \sum_{X_2} x_i \mathbb{P}_{X_2}(x_i) = (0 \times \frac{1}{8}) + (1 \times \frac{3}{8}) + (2 \times \frac{3}{8}) + (3 \times \frac{1}{8}) = 1.5$$

| $X_1X_2$ | 0 | 1 | 2 | 3 | 4 | 6 | 9 |
|---|---|---|---|---|---|---|---|
| $\mathbb{P}_{X_1,X_2}(x_{i1}, x_{i2})$ | $\frac{15}{64}$ | $\frac{9}{64}$ | $\frac{9}{64}+\frac{9}{64}$ | $\frac{3}{64}+\frac{3}{64}$ | $\frac{9}{64}$ | $\frac{3}{64}+\frac{3}{64}$ | $\frac{1}{64}$ |

$$\mathbb{E}(X_1X_2) = \sum_{X_1,X_2} x_{i1}x_{i2}\mathbb{P}_{X_1,X_2}(x_{i1}, x_{i2}) =$$
$$(0 \times \frac{1}{64}) + (1 \times \frac{9}{64}) + (2 \times \frac{18}{64}) + (3 \times \frac{6}{64}) + (4 \times \frac{9}{64}) + (6 \times \frac{6}{64}) + (9 \times \frac{1}{64}) = 2.25$$

$$Cov(X, Y) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = 2.25 - (1.5 \times 1.5) = 0 \Rightarrow Corr(X, Y) = 0$$

- If X and Y are independent $\Rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \Rightarrow Cov(X, Y) = Corr = 0.$

# Training-Random Variables

Example: (Zero covariance does not imply independence.)
Let X be a random variable that takes values $-2, -1, 0, 1, 2$; each with probability $\frac{1}{5}$.

| X | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $\mathbb{P}_X(x)$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |

Let $Y = X^2$. So we get the following joint PMF

| $X/Y = X^2$ | 0 | 1 | 4 | $\mathbb{P}(X = k)$ |
|---|---|---|---|---|
| -2 | 0 | 0 | $\frac{1}{5}$ | $\frac{1}{5}$ |
| -1 | 0 | $\frac{1}{5}$ | 0 | $\frac{1}{5}$ |
| 0 | $\frac{1}{5}$ | 0 | 0 | $\frac{1}{5}$ |
| 1 | 0 | $\frac{1}{5}$ | 0 | $\frac{1}{5}$ |
| 2 | 0 | 0 | $\frac{1}{5}$ | $\frac{1}{5}$ |
| $\mathbb{P}(Y = k)$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{2}{5}$ | 1 |

- Show that $Cov(X, Y) = 0$ but X and Y are not independent.

# Training-Random Variables

$\mu_X = \mathbb{E}(X) = \sum_{x_i} x_i \mathbb{P}(X = x_i) = (-2 \times \frac{1}{5}) + (-1 \times \frac{1}{5}) + (0 \times \frac{1}{5}) + (1 \times \frac{1}{5}) + (2 \times \frac{1}{5}) = 0$

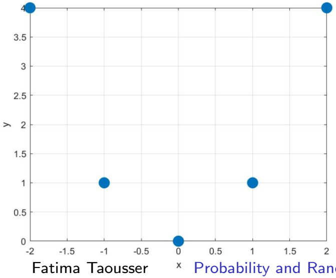
$\mu_Y = \mathbb{E}(Y) = \sum_{y_i} y_i \mathbb{P}(Y = y_i) = (0 \times \frac{1}{5}) + (1 \times \frac{2}{5}) + (4 \times \frac{2}{5}) = 2$

- Let us compute $\mathbb{E}(XY)$:

| XY | -8 | -4 | -2 | -1 | 0 | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{P}(XY)$ | $\frac{1}{5}$ | 0 | 0 | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | 0 | 0 | $\frac{1}{5}$ |

$\mathbb{E}(XY) = (-8 \times \frac{1}{5}) + (-4 \times 0) + (-2 \times 0) + (-1 \times \frac{1}{5}) + (0 \times \frac{1}{5}) + (1 \times \frac{1}{5}) + (2 \times 0) + (4 \times 0) + (8 \times \frac{1}{5}) = 0$

$\Rightarrow \; Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0 - (0 \times 2) = 0 \Rightarrow corr = 0 \rightarrow$ Since the relationship between data is not linear



# Multivariate Random Variables

- **The population correlation:** is an index expressing the degree of association between two measured variables for a complete population of interest.

$$Cov(X,Y) = \frac{\sum_{(X,Y)}(x-\mu_X)(y-\mu_Y)}{N}, \quad Corr = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{1}{N}\frac{\sum_{(X,Y)}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}$$

where $N$ is the number of data values of all the population.

- **The correlation coefficient for the sample:** If we had data for the entire population, we could find the population correlation coefficient. But if we have only sample data, we cannot calculate the population correlation coefficient $\rightarrow$ Our estimate of the unknown population correlation coefficient is done by the sample correlation.

$$Cov(X,Y) = \frac{\sum_{(X,Y)}(x-\mu_X)(y-\mu_Y)}{n-1}, \quad Corr = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{1}{n-1}\frac{\sum_{(X,Y)}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}$$

where $n$ is the number of data values of the sample set.

- **For example:** A researcher could obtain income and education information for all families in a town and calculate a population correlation coefficient for the entire town. In contrast, the sample correlation coefficient indexes the correlation for a sample of those cases (e.g., every fourth family from a list of all those in the town).

# Multivariate Random Variables

- Example1: Let the following random variables (sets of data):

| X | Y |
|---|---|
| 1 | 5 |
| 2 | 9 |
| 3 | 14 |
| 4 | 13 |
| 5 | 16 |

$\Rightarrow$

| X Y |
|---|
| 5 |
| 18 |
| 42 |
| 52 |
| 80 |

- Find the expectation and the variance of each random variable, and find the covariance and the correlation of X and Y

- Solution:

$$\mu_X = \mathbb{E}(X) = \frac{(1+2+3+4+5)}{5} = 3, \quad \mu_Y = \mathbb{E}(Y) = \frac{(5+9+14+13+16)}{5} = 11.4$$

$$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{(1^2+2^2+3^2+4^2+5^2)}{5} - 3^2 = 2 \rightarrow \sigma_X = \sqrt{2}$$

$$V(Y) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{(5^2+9^2+14^2+13^2+16^2)}{5} - (11.4)^2 = 15.44 \rightarrow \sigma_Y = \sqrt{15.44}$$

# Multivariate Random Variables

$$Cov(X,Y) = \frac{\sum_{(X,Y)}(x-\mu_X)(y-\mu_Y)}{N} = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$= \frac{(5+18+42+52+80)}{5} - (3 \times 11.4) = 5.2$$

$$Corr = \rho = \frac{1}{N}\frac{\sum_{(X,Y)}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{5.2}{\sqrt{2 \times 15.44}} = 0.935$$

# Multivariate Random Variables

- **Matlab code**

x=[1,2,3,4,5]   y=[5,9,14,13,16]

C=cov(x,y), $E_1 = mean(x)$, $E_2 = mean(y)$, $V_1 = var(x, 1)$, $V_2 = var(y, 1)$

R=corrcoef(x,y)

C=5.2 ,   R=0.935

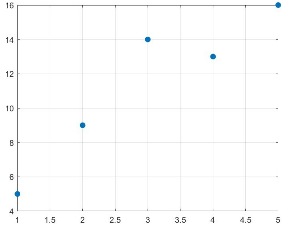
- **If we will scale X by 10 and Y by 2**, we will get

x=10*[1,2,3,4,5]   y=2*[5,9,14,13,16]

C=cov(x,y,1),

R=corrcoef(x,y)

C=104,   R=0.935



# Multivariate Random Variables

- Example2: Let the collected data on the house prices according to the size of feet$^2$ :

| Size in feet$^2$($X$) | prices (\$) in k (Y) |
|---|---|
| 2104 | 460 |
| 1416 | 232 |
| 1534 | 315 |
| 852 | 178 |
| 1416 | 310 |
| 2300 | 280 |
| 999 | 290 |
| 790 | 250 |
| 605 | 210 |
| 601 | 150 |
| 400 | 100 |

- Find the Covariance and the correlation of the random variables X and Y

# Multivariate Random Variables

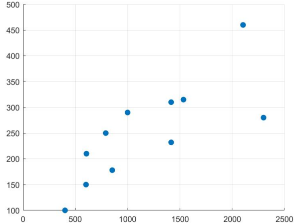
- Example2-Solution:

Using the Matlab code we get:

$$\mu_X = 1.1834.10^3, \quad \mu_Y = 252.272. \quad V(X) = 3.55.10^5, \quad V(Y) = 8543.3,$$
$$Cov(X,Y) = 0.4316.10^5, \quad Corr(X,Y) = 0.783$$

- **Matlab code for the plot:**

```
x=[2104, 1416, 1534, 852, 1416, 2300, 999, 790, 605, 601,400]
y=[460, 232, 315, 178, 310, 280, 290, 250, 210, 150, 100]
sz=50
scatter(x,y,sz,'filled')
```



# Multivariate Random Variables

## - Limitations of Correlation

Like other aspects of statistical analysis, correlation can be misinterpreted:

- Small sample sizes may yield unreliable results, even if it appears as though correlation between two variables is strong. Alternatively, a small sample size may yield uncorrelated findings when the two variables are in fact linked.
- Correlation only shows how one variable is connected to another and may not clearly identify how a single instance or outcome can impact the correlation coefficient.
- Correlation may also be misinterpreted if the relationship between two variables is nonlinear. It is much easier to identify two variables with a positive or negative correlation. However, two variables may still be correlated with a more complex relationship.

# Training-Random Variables

- Example2:

Roll two fair dice and denote the outcome $(X, Y) \rightarrow 36$ elements.
Let $Z_1 = X + Y$ and $Z_2 = \max\{X, Y\}$. Find $Corr(Z_1, Z_2)$.

$$Z_1 = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}, \quad Z_2 = \{1, 2, 3, 4, 5, 6\} \rightarrow \text{The joint PMF is}$$

| $Z_2/Z_1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{1}{36}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | $\frac{2}{36}$ | $\frac{1}{36}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | $\frac{2}{26}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

# Training-Random Variables

| $Z_2/Z_1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $\mathbb{P}(Z_2)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{1}{36}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{36}$ |
| 2 | 0 | $\frac{2}{36}$ | $\frac{1}{36}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{3}{36}$ |
| 3 | 0 | 0 | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{5}{36}$ |
| 4 | 0 | 0 | 0 | $\frac{2}{26}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 0 | 0 | 0 | 0 | $\frac{7}{36}$ |
| 5 | 0 | 0 | 0 | 0 | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 0 | 0 | $\frac{9}{36}$ |
| 6 | 0 | 0 | 0 | 0 | 0 | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | $\frac{11}{36}$ |
| $\mathbb{P}(Z_1)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | 1 |

# Training-Random Variables

$$\mu_{Z_1} = \mathbb{E}(Z_1) = \sum_{z_i} z_i \mathbb{P}(Z_1 = z_i) = \frac{2+6+12+20+30+42+40+36+30+22+12}{36} = 7$$

$$\mu_{Z_2} = \mathbb{E}(Z_2) = \sum_{z_i} z_i \mathbb{P}(Z_2 = z_i) = \frac{1+6+15+28+45+66}{36} = 4.47$$

$$V(Z_1) = \mathbb{E}(Z_1^2) - (\mathbb{E}(Z_1))^2 = \sum_{z_i} z_i^2 \mathbb{P}(Z_1 = z_i) - 7^2$$

$$= \frac{(4+18+48+100+180+294+320+324+300+242+144)}{36} - 7^2 = 5.83 \rightarrow \sigma(Z_1) = 2.41$$

$$V(Z_2) = \mathbb{E}(Z_2^2) - (\mathbb{E}(Z_2))^2 = \sum_{z_i} z_i^2 \mathbb{P}(Z_2 = z_i) - (4.47)^2 =$$
$$\frac{(1+12+45+112+225+396)}{36} - 4.47^2 = 1.99 \rightarrow \sigma(Z_2) = 1.41$$

$$\mathbb{E}(Z_1 Z_2) = \sum_{z_i} z_i \mathbb{P}(Z_1 Z_2 = z_i) = 33.88 \Rightarrow Cov(Z_1, Z_2) = \mathbb{E}(Z_1 Z_2) - \mathbb{E}(Z_1)\mathbb{E}(Z_2) = 2.59$$

$$Corr(Z_1, Z_2) = \frac{Cov(X, Y)}{\sigma(Z_1)\sigma(Z_2)} = \frac{2.59}{2.41 \times 1.41} = 0.76$$

# Training-Random Variables

- Example:

Let's say you are the new owner of a small ice-cream shop in a little village near the beach. You noticed that there was more business in the warmer months than the cooler months. Before you alter your purchasing pattern to match this trend, you want to be sure that the relationship is real.
Let's work through these two statistical measures using the data that you collected when looking for trends with your ice cream shop.

| Temperature (X) | Number of customers (Y) |
|---|---|
| 98 | 15 |
| 87 | 12 |
| 90 | 10 |
| 85 | 10 |
| 95 | 16 |
| 75 | 7 |

- Compute the covariance and the correlation

# Training-Random Variables

| Temperature (X) | Number of customers (Y) | $x_i - \mu_X$ | $y_i - \mu_Y$ | $(x_i - \mu_X)(y_i - \mu_Y)$ |
|---|---|---|---|---|
| 98 | 15 | 9.67 | 3.34 | 32.29 |
| 87 | 12 | -1.33 | 0.34 | -0.45 |
| 90 | 10 | 1.67 | -1.66 | -2.77 |
| 85 | 10 | -3.33 | -1.66 | 5.52 |
| 95 | 16 | 6.67 | 4.34 | 28.94 |
| 75 | 7 | -13.33 | -4.66 | 62.11 |

$$\mu_X = \mathbb{E}(X) = \frac{\sum x_i}{6} = 88.33, \qquad \mu_Y = \mathbb{E}(Y) = \frac{\sum y_i}{6} = 11.66$$

$$V(X) = \frac{\sum (x_i - \mu_X)^2}{6} = 55.22 \rightarrow \sigma_X = 7.43, \qquad V(Y) = \frac{\sum (y_i - \mu_Y)^2}{6} = 9.55 \rightarrow \sigma_X = 3.09$$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_X)(y_i - \mu_Y)}{6} = 20.94$$

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{20.94}{7.43 \times 3.09} = 0.91$$

# Training-Random Variables

**- Method 2:**

| Temperature (X) | Number of customers (Y) | $X^2$ | $Y^2$ | $XY$ |
|---|---|---|---|---|
| 98 | 15 | 9604 | 225 | 1470 |
| 87 | 12 | 7569 | 144 | 1044 |
| 90 | 10 | 8100 | 100 | 900 |
| 85 | 10 | 7225 | 100 | 850 |
| 95 | 16 | 9025 | 256 | 1520 |
| 75 | 7 | 5625 | 49 | 525 |

$$\mathbb{E}(XY) = \frac{\sum x_i y_i}{6} = 10501.5$$

$$V(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = 7858 - (88.33)^2 = 55.22 \rightarrow \sigma_X = 7.43$$

$$V(Y) = \mathbb{E}(Y^2) - \mathbb{E}^2(Y) = 145.6667 - (11.66)^2 = 9.55 \rightarrow \sigma_X = 3.09$$

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 10501.5 - (88.33 * 11.66) = 21.75$$

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{21.75}{7.43 \times 3.09} = 0.9474$$

# Continuous Multi Random Variables

- Continuous multi-random variables:

▷ Joint density function: $f_{X,Y}(x,y) \geq 0 \rightarrow \int \int_S f_{X,Y}(x,y)\ dx\ dy = 1$

▷ Marginal density function: $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\ dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\ dx$

▷ Probability: $P(X \leq x_i; Y \leq y_i) = \int_{-\infty}^{y_i} \int_{-\infty}^{x_i} f_{X,Y}(x,y)\ dx\ dy$

▷ Conditional probability: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

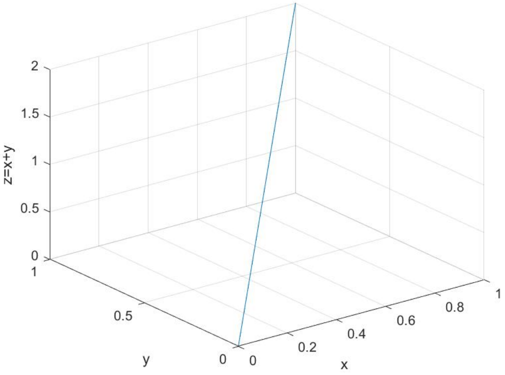
▷ Independence: $f_{X,Y}(x,y) = f_X(x).f_Y(y)$ or $f_{X|Y}(x|y) = f_X(x)$

▷ Expectation: $\mathbb{E}[g(x,y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y)\ dx\ dy$

# Continuous Multi Random Variables

- Example: Let the joint probability density function (PDF)

$$f_{X,Y}(x,y) = \begin{cases} x+y; & 0 < x, y \le 1 \\ 0; & \text{Otherwise} \end{cases}$$



# Continuous Multi Random Variables

$$f_{X,Y}(x,y) = \begin{cases} x+y; & 0 < x, y \le 1 \\ 0; & \text{Otherwise} \end{cases}$$

- It is a density function since

$$\int_0^1 \int_0^1 f_{X,Y}(x,y)\,dx\,dy = \int_0^1 \int_0^1 (x+y)\,dx\,dy = \int_0^1 [\int_0^1 (x+y)\,dx]\,dy = \int_0^1 [\frac{1}{2}x^2 + yx]_0^1\,dy$$

$$= \int_0^1 (\frac{1}{2} + y)\,dy = [\frac{1}{2}y + \frac{1}{2}y^2]_0^1 = 1$$

- The marginal probabilities are given by

$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 (x+y)\,dy = [xy + \frac{1}{2}y^2]_0^1 = x + \frac{1}{2}$

$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx = \int_0^1 (x+y)\,dx = [\frac{1}{2}x^2 + xy]_0^1 = y + \frac{1}{2}$

- Note that: $f_X(x).f_Y(y) = (x + \frac{1}{2})(y + \frac{1}{2}) \neq x + y \Rightarrow$ X and Y are not independent

# Continuous Multi Random Variables

- $\mathbb{P}(X \leq 0.25; Y \leq 0.75) = \int_0^{0.75}\int_0^{0.25} f_{X,Y}(x,y)\,dx\,dy = \int_0^{0.75}\int_0^{0.25}(x+y)\,dx\,dy = \int_0^{0.75}[\int_0^{0.2}$

$$= \int_0^{0.75}[\frac{1}{2}x^2 + yx]_0^{0.25}\,dy = \int_0^{0.75}(0.0312 + 0.25y)\,dy = [0.0312y + \frac{0.25}{2}y^2]_0^{0.75} = 0.0937$$

- $\mathbb{E}[X] = \int_0^1\int_0^1 x f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 x(x+y)\,dx\,dy = \int_0^1[\int_0^1 x^2 + xy\,dx]\,dy$

$$= \int_0^1[\frac{1}{3}x^3 + \frac{1}{2}x^2y]_0^1\,dy = \int_0^1(\frac{1}{3} + \frac{1}{2}y)\,dy = [\frac{1}{3}y + \frac{1}{4}y^2]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} = \mathbb{E}[Y]$$

- $\mathbb{E}[X^2] = \int_0^1\int_0^1 x^2 f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 x^2(x+y)\,dx\,dy = \int_0^1[\int_0^1 x^3 + x^2y\,dx]\,dy$

$$= \int_0^1[\frac{1}{4}x^4 + \frac{1}{3}x^3y]_0^1\,dy = \int_0^1(\frac{1}{4} + \frac{1}{3}y)\,dy = [\frac{1}{4}y + \frac{1}{6}y^2]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{10}{24} = \mathbb{E}[Y^2]$$

# Continuous Multi Random Variables

- $V(x) = \mathbb{E}[X^2] - \mathbb{E}^2[X] = \frac{10}{24} - (\frac{7}{12})^2 = 0.076$

- $\mathbb{E}[X - 2Y] = \int_0^1 \int_0^1 (x - 2y) f_{X,Y}(x, y)\ dx\ dy = \int_0^1 \int_0^1 (x - 2y)(x + y)\ dx\ dy$

$$= \int_0^1 [\int_0^1 (x^2 - yx - 2y^2)\ dx]\ dy = \int_0^1 [\frac{1}{3}x^3 + \frac{1}{2}yx^2 - 2y^2x]_0^1\ dy$$

$$= \int_0^1 (\frac{1}{3} + \frac{1}{2}y - 2y^2)\ dy = [\frac{1}{3}y - \frac{1}{3}y^3 - \frac{2}{3}y^3]_0^1 = \frac{-2}{3}$$

# Continuous Multi Random Variables

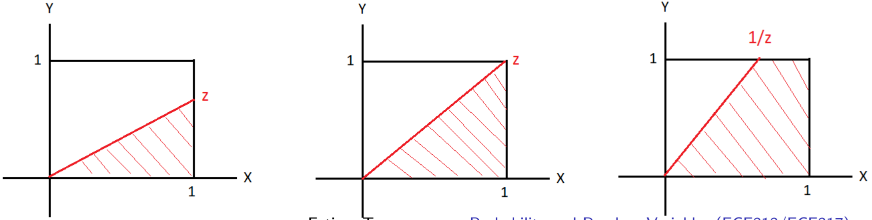
Example 2:

Consider the two independent random variables X and Y that follow an uniform distribution on $[0,1]$. So we have: $f_X(x) = \begin{cases} 1; & x \in [0,1] \\ 0; & \text{Otherwise} \end{cases}$ , $f_Y(y) = \begin{cases} 1; & y \in [0,1] \\ 0; & \text{Otherwise} \end{cases}$

$\Rightarrow f_{X,Y}(x,y) = f_X(x).f_Y(y) = \begin{cases} 1; & 0 \leq x; y \leq 1 \\ 0; & \text{Otherwise} \end{cases}$

- Let $Z = \dfrac{Y}{X}$. What is the density function of $Z$?

• $Z = \dfrac{Y}{X} \Rightarrow Y = Z.X \rightarrow Z$ is the slop. $F(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(Y \leq zX) \rightarrow$ the area under the line $Y = zX$. There are 3 cases:



# Continuous Multi Random Variables

- Case 1: $0 \leq z < 1 \rightarrow F_Z(z) = \dfrac{z}{2} \rightarrow$ The area under the line.
- Case 2: $z = 1 \rightarrow F_Z(z) = \dfrac{1}{2} \rightarrow$ The area under the line.
- Case 3: $z > 1 \rightarrow F_Z(z) = 1 - \dfrac{1}{2z} \rightarrow$ The area under the line.
- $F_Z(z) = \begin{cases} \dfrac{1}{2}; & 0 \leq z \leq 1 \\ 1 - \dfrac{1}{2z}; & z > 1 \end{cases} \Rightarrow f_Z(z) = \dfrac{dF_Z(z)}{dz} = \begin{cases} \dfrac{1}{2}; & 0 \leq z \leq 1 \\ \dfrac{1}{2z^2}; & z > 1 \end{cases}$